Finsleroid-Space Supplemented by Angle

G.S. Asanov

Division of Theoretical Physics, Moscow State University
119992 Moscow, Russia
(e-mail: asanov@newmail.ru)

Abstract

Our previous exploration of the $\mathcal{E}_g^{PD}$-geometry has shown that the field is promising. Namely, the $\mathcal{E}_g^{PD}$-approach is amenable to development of novel trends in relativistic and metric differential geometry and can particularly be effective in context of the Finslerian or Minkowskian Geometries. The main point of the present paper is the tenet that the $\mathcal{E}_g^{PD}$-space-associated one-vector Finslerian metric function admits in quite a natural way an attractive two-vector extension, thereby giving rise to angle and scalar product. The underlying idea is to derive the angular measure from the solutions to the geodesic equation, which prove to be obtainable in an explicit simple form. The respective investigation is presented in Part I. Part II serves as an extended Addendum enclosing the material which is primary for the $\mathcal{E}_g^{PD}$-space. The Finsleroid, instead of the unit sphere, is taken now as carrier proper of the spherical image. The indicatrix is, of course, our primary tool.

INTRODUCTION

Various known attempts to introduce the concept of angle in the Minkowskian or Finslerian spaces [1-8] were steadily encountered with drawback positions:

"Therefore no particular angular measure can be entirely natural in Minkowski geometry. This is evidenced by the innumerable attempts to define such a measure, none of which found general acceptance". (Busemann [2], p. 279.)

"Unfortunately, there exists a number of distinct invariants in a Minkowskian space all of which reduce to the same classical euclidean invariant if the Minkowskian space degenerates into a euclidean space. Consequently, distinct definitions of the trigonometric functions and of angles have appeared in the literature concerning Minkowskian and Finsler spaces". (Rund [3], p. 26)

The fact that the attempts have never been unambiguous seems to be due to a lack of the proper tools. The opinion was taken for granted that the angle ought to be defined or constructed in terms of the basic Finslerian metric tensor (and whence ought to be explicated from the initial Finslerian metric function). Let us doubt the opinion from the very beginning. Instead, we would like to raise alternatively the tenet that the angle is a concomitant of the geodesics (and not of the metric proper). The angle is determined by two vectors (instead of one vector in case of the length) and actually implies using a due extension of the Finslerian metric function to a two-vector metric function (the scalar product). Below, we apply this tenet to study the $\mathcal{E}_g^{PD}$-spaces [9-12] in which certainly one needs to use not only length but also angle and scalar product.

Accordingly, in Part I, we first deal with the geodesic equation (Sec. 1.1). Remarkably, the equation admits a simple and explicit general solution. After that, the angle between two vectors is explicated. It is commonly expected that the angular measure should be additive for angles with the same vertex. Remarkably, the angle found is a factor of the euclidean angle and, therefore, is additive. The Cosine Theorem remains valid if the euclidean angle is replaced by the angle found. The respective scalar product is obtained.

In Sec. 1.2, we introduce the associated two-vector metric tensor and demonstrate that at the equality of vectors the tensor reduces exactly to the one-vector Finslerian metric tensor of the $\mathcal{E}_g^{PD}$-space. The concomitant two-vector metric tensor is given by the components (2.2). The orthonormal frame thereto is also found in a lucid explicit form. After that, in Sec. 1.3, the possibility of converting the theory into the co-approach is presented, and in Sec. 1.4 the $\mathcal{E}_g^{PD}$-extension of the parallelogram law of vector addition is derived; it occurs possible to find the sum vector and the difference vector in a nearest approximation. Part I ends with Sec. 1.5 in which we return the treatment from the auxiliary quasi-euclidean framework to the primary $\mathcal{E}_g^{PD}$-approach.

Part II reviews the fundamental ingredients of the $\mathcal{E}_g^{PD}$-space in great detail.

PART I:

$\mathcal{E}_g^{PD}$-SPACE GEODESICS, ANGLE, SCALAR PRODUCT

1.1. Derivation of geodesics and angle in associated quasi-euclidean space

For the space under study, the geodesics should be obtained as solutions to the equation

$$\frac{d^2R^p}{ds^2} + C_q{}^p{}_r(g;R)\frac{\partial R^q}{\partial ds}\frac{\partial R^r}{\partial ds} = 0 \tag{1.1}$$

which coefficients ${C_p}^q{}_r$ are given by the list placed the end of Sec. 2.2 of Part II. To avoid complications of calculations involved, it proves convenient to transfer the consideration in the quasi-euclidean approach (see Sec. 2.3 of Part II). Accordingly, we put

$$\sqrt{g_{pq}(g;R)dR^pdR^q} = \sqrt{n_{pq}(g;t)dt^pdt^q} \tag{1.2}$$

and

$$R^p(s) = \mu^p(g;t^r(s)) \tag{1.3}$$

together with

$$\frac{dR^p(s)}{ds} = \mu^p_q(g;t^r(s))\frac{dt^q(s)}{ds}, \tag{1.4}$$

where $\mu^p(g;t^r)$ and $\mu^p_q(g;t^r)$ are the coefficients given, respectively by Eqs. (3.14) and (3.38)-(3.40) of Part II. Let a curve C: $t^p = t^p(s)$ be given in the quasi-euclidean space, with *the arc-length parameter* s along the curve being defined by the help of the differential

$$ds = \sqrt{n_{pq}(g;t)dt^pdt^q}, \tag{1.5}$$

where $n_{pq}(g;t)$ is the associated quasi-euclidean metric tensor given by Eq. (3.49) in Part II. Respectively, *the tangent vectors*

$$u^p = \frac{dt^p}{ds} \tag{1.6}$$

to the curve are unit, in the sense that

$$n_{pq}(g;t)u^pu^q = 1. \tag{1.7}$$

Since $L_p = \partial S/\partial t^p$, we have

$$L_pu^p = \frac{dS}{ds}. \tag{1.8}$$

Here, $S^2(t) = n_{pq}(g;t)t^pt^q = r_{pq}t^pt^q$ (see Eq. (3.46) in Part II). Using Eq. (4.16) of Part II leads through well-known arguments (see, e.g., [13-14]) to the following equation of geodesics in the quasi-euclidean space:

$$\frac{d^2\mathbf{t}}{ds^2} = \frac{1}{4}G^2\frac{\mathbf{t}}{S^2}H_{pq}u^pu^q, \tag{1.9}$$

where $H_{pq} = h^2(n_{pq} - L_pL_q)$ (see Eq. (4.4) in Part II) and $\mathbf{t} = \{t^p\}$. We obtain

$$\frac{d^2\mathbf{t}}{ds^2} = \frac{1}{4}g^2\frac{\mathbf{t}}{S^2}\left(1-(\frac{dS}{ds})^2\right) = \frac{1}{4}g^2(a^2-b^2)\frac{\mathbf{t}}{S^4} \tag{1.10}$$

and

$$\frac{d^2\mathbf{t}}{ds^2} = \frac{1}{4}g^2(a^2-b^2)\frac{\mathbf{t}}{S^4} \tag{1.11}$$

with

$$S^2(s) = a^2 + 2bs + s^2, \tag{1.12}$$

where a and b are two constants of integration.

If we put

$$S(\Delta s) = \sqrt{a^2 + 2b\Delta s + (\Delta s)^2} \tag{1.13}$$

and

$$\mathbf{t}_1 = \mathbf{t}(0), \qquad \mathbf{t}_2 = \mathbf{t}(\Delta s), \tag{1.14}$$

then we get

$$a = \sqrt{(\mathbf{t}_1\mathbf{t}_1)} \tag{1.15}$$

and

$$S(\Delta s) = \sqrt{(\mathbf{t}_2\mathbf{t}_2)} \tag{1.16}$$

together with

$$(\mathbf{t}_1\mathbf{t}_2) = aS(\Delta s)\cos\Big[h \arctan\frac{\sqrt{a^2-b^2}\,\Delta s}{a^2+b\Delta s}\Big]. \tag{1.17}$$

Here, $\mathbf{t}_1$ and $\mathbf{t}_2$ are two vectors with the fixed origin O; they point to the beginning of the geodesic and to the end of the geodesic, respectively. The notation parenthesis couple (..) is used for the euclidean scalar product, so that $(\mathbf{t}_1\mathbf{t}_1) = r_{pq}t_1^p t_1^q$, $(\mathbf{t}_1\mathbf{t}_2) = r_{pq}t_1^p t_2^q$, and r_{pq} is a euclidean metric tensor; $r_{pq} = \delta_{pq}$ in case of orthogonal basis; δ stands for the Kronecker symbol. From (1.15)-(1.17) it directly follows that

$$\frac{\sqrt{a^2-b^2}\,\Delta s}{a^2+b\Delta s} = \tan\Big[\frac{1}{h}\arccos\frac{(\mathbf{t}_1\mathbf{t}_2)}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\,\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}\Big]. \tag{1.18}$$

The equality (1.18) suggests the idea to introduce

DEFINITION. *The $\mathcal{E}_g^{PD}$-associated angle* is given by

$$\alpha \stackrel{\text{def}}{=} \frac{1}{h}\arccos\frac{(\mathbf{t}_1\mathbf{t}_2)}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\,\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}, \tag{1.19}$$

so that

$$\alpha = \frac{1}{h}\alpha_{euclidean}. \tag{1.20}$$

Such an angle is obviously *additive*:

$$\alpha(\mathbf{t}_1,\mathbf{t}_3) = \alpha(\mathbf{t}_1,\mathbf{t}_2) + \alpha(\mathbf{t}_2,\mathbf{t}_3). \tag{1.21}$$

Also,

$$\alpha(\mathbf{t},\mathbf{t}) = 0. \tag{1.22}$$

With the angle (1.19), we ought to propose

DEFINITION. Given two vectors $\mathbf{t}_1$ and $\mathbf{t}_2$, we say that the vectors are $\mathcal{E}_g^{PD}$-*perpendicular,* if

$$\cos\left(\alpha(\mathbf{t}_1,\mathbf{t}_2)\right) = 0. \tag{1.23}$$

Since the vanishing (1.23) implies

$$\alpha_{quasi-euclidean}(\mathbf{t}_1,\mathbf{t}_2) = \frac{\pi}{2}, \tag{1.24}$$

in view of 1.20) we ought to conclude that

$$\alpha_{euclidean}(\mathbf{t}_1,\mathbf{t}_2) = \frac{\pi}{2}h \le \frac{\pi}{2}. \tag{1.25}$$

Therefore, vectors perpendicular in the quasi-euclidean sense proper look like acute vectors as observed from associated euclidean standpoint.

With the equality

$$(\sqrt{a^2-b^2}\,\Delta s)^2 + (a^2 + b\Delta s)^2 \equiv a^2 S^2(\Delta s), \tag{1.26}$$

we also establish the relations

$$\sqrt{a^2-b^2}\,\Delta s = aS(\Delta s)\sin\alpha \tag{1.27}$$

and

$$a^2 + b\Delta s = aS(\Delta s)\cos\alpha. \tag{1.28}$$

They entail the equality

$$\frac{b}{\sqrt{a^2-b^2}} = \frac{S(\Delta s)\cos\alpha - a}{S(\Delta s)\sin\alpha} \tag{1.29}$$

from which the quantity b can be explicated.

Thus *each member of the involved set* $\{a, b, \Delta s, S(\Delta s)\}$ *can be explicitly expressed through the input vectors* $\mathbf{t}_1$ *and* $\mathbf{t}_2$. For many cases it is worth rewriting the equality (1.24) as

$$S^2(\Delta s) = (\Delta s)^2 - a^2 + 2(a^2 + b\Delta s). \tag{1.30}$$

Thus we have arrived at the following substantive items:

The $\mathcal{E}_g^{PD}$*-Case Cosine Theorem*

$$(\Delta s)^2 = S^2(\Delta s) + a^2 - 2aS(\Delta s)\cos\alpha\,; \tag{1.31}$$

The $\mathcal{E}_g^{PD}$*-Case Two-Point Length*

$$(\Delta s)^2 = (\mathbf{t}_1\mathbf{t}_1) + (\mathbf{t}_2\mathbf{t}_2) - 2\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\,\sqrt{(\mathbf{t}_2\mathbf{t}_2)}\cos\alpha\,; \tag{1.32}$$

The $\mathcal{E}_g^{PD}$*-Case Scalar Product*

$$< \mathbf{t}_1, \mathbf{t}_2 >= \sqrt{(\mathbf{t}_1\mathbf{t}_1)}\,\sqrt{(\mathbf{t}_2\mathbf{t}_2)}\cos\alpha\,; \tag{1.33}$$

The $\mathcal{E}_g^{PD}$*-Case Perpendicularity*

$$< \mathbf{t}_1, \mathbf{t}_2 >= \sqrt{(\mathbf{t}_1\mathbf{t}_1)}\,\sqrt{(\mathbf{t}_2\mathbf{t}_2)}. \tag{1.34}$$

The identification

$$|\mathbf{t}_2 \ominus \mathbf{t}_1|^2 = (\Delta s)^2 \tag{1.35}$$

yields another lucid representation

$$|\mathbf{t}_2 \ominus \mathbf{t}_1|^2 = (\mathbf{t}_1\mathbf{t}_1) + (\mathbf{t}_2\mathbf{t}_2) - 2\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\,\sqrt{(\mathbf{t}_2\mathbf{t}_2)}\cos\alpha\,. \tag{1.36}$$

The consideration can be completed by

THEOREM. *A general solution to the geodesic equation* (1.11) *can explicitly be found as follows:*

$$\mathbf{t}(s) =$$

$$= \frac{S(s)}{a} \frac{\sin\left[h \arctan \dfrac{\sqrt{a^2-b^2}\,(\Delta s - s)}{a^2 + b\Delta s + (b+\Delta s)s}\right]}{\sin\left[h \arctan \dfrac{\sqrt{a^2-b^2}\,\Delta s}{a^2+b\Delta s}\right]} \mathbf{t}_1 + \frac{S(s)}{S(\Delta s)} \frac{\sin\left[h \arctan \dfrac{\sqrt{a^2-b^2}\,s}{a^2+bs}\right]}{\sin\left[h \arctan \dfrac{\sqrt{a^2-b^2}\,\Delta s}{a^2+b\Delta s}\right]} \mathbf{t}_2. \tag{1.37}$$

The euclidean limit proper is

$$\mathbf{t}(s)\Big|_{g=0} = \frac{(\Delta s - s)\mathbf{t}_1 + s\mathbf{t}_2}{\Delta s} = \mathbf{t}_1 + (\mathbf{t}_2 - \mathbf{t}_1)\frac{s}{\Delta s},$$

so that the geodesics become straight. From (1.35) the equality

$$(\mathbf{t}(s)\mathbf{t}(s)) = S^2(s) \tag{1.38}$$

follows, in agreement with (1.12). Since the general solution (1.35) is such that the right-hand side is spanned by two fixed vectors, $\mathbf{t}_1$ and $\mathbf{t}_2$, we are entitled concluding that *the geodesics under study are plane curves.*

Calculating the first derivative yields simply the formula

$$\frac{d\mathbf{t}}{ds}(s) = \frac{b+s}{S^2(s)}\mathbf{t} - \frac{\sqrt{a^2-b^2}\,h}{aS(s)} \frac{\cos\left[h \arctan \dfrac{\sqrt{a^2-b^2}\,(\Delta s - s)}{a^2 + b\Delta s + (b+\Delta s)s}\right]}{\sin\left[h \arctan \dfrac{\sqrt{a^2-b^2}\,\Delta s}{a^2+b\Delta s}\right]} \mathbf{t}_1$$

$$+ \frac{\sqrt{a^2-b^2}\,h}{S(s)S(\Delta s)} \frac{\cos\left[h \arctan \dfrac{\sqrt{a^2-b^2}\,s}{a^2+bs}\right]}{\sin\left[h \arctan \dfrac{\sqrt{a^2-b^2}\,\Delta s}{a^2+b\Delta s}\right]} \mathbf{t}_2, \tag{1.39}$$

in which the representation (1.35) should be inserted. The right-hand part here is such that

$$\mathbf{t}(s)\left(\frac{d\mathbf{t}}{ds}(s) - \frac{b+s}{S^2(s)}\mathbf{t}\right) = 0,$$

from which observation the useful equality

$$\mathbf{t}(s)\frac{d\mathbf{t}}{ds}(s) = b + s \tag{1.40}$$

ensues.

For the vectors

$$\mathbf{b}_1 \stackrel{\text{def}}{=} \frac{1}{2}\frac{\partial|\mathbf{t}_2 \ominus \mathbf{t}_1|^2}{\partial \mathbf{t}_1}, \qquad \mathbf{b}_2 \stackrel{\text{def}}{=} \frac{1}{2}\frac{\partial|\mathbf{t}_2 \ominus \mathbf{t}_1|^2}{\partial \mathbf{t}_2}, \tag{1.41}$$

we can obtain the simple representations

$$\mathbf{b}_1 = \mathbf{t}_1 - \frac{\mathbf{t}_1}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}}\sqrt{(\mathbf{t}_2\mathbf{t}_2)}\,\cos\alpha - \frac{\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}{h\sqrt{(\mathbf{t}_1\mathbf{t}_1)}}\mathbf{d}_1\sin\alpha \tag{1.42}$$

and

$$\mathbf{b}_2 = \mathbf{t}_2 - \frac{\mathbf{t}_2}{\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\,\cos\alpha - \frac{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}}{h\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}\mathbf{d}_2\sin\alpha, \tag{1.43}$$

where the convenient vectors

$$\mathbf{d}_1 = \frac{(\mathbf{t}_1\mathbf{t}_1)\mathbf{t}_2 - (\mathbf{t}_1\mathbf{t}_2)\mathbf{t}_1}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2) - (\mathbf{t}_1\mathbf{t}_2)^2}}, \qquad \mathbf{d}_2 = \frac{(\mathbf{t}_2\mathbf{t}_2)\mathbf{t}_1 - (\mathbf{t}_1\mathbf{t}_2)\mathbf{t}_2}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2) - (\mathbf{t}_1\mathbf{t}_2)^2}} \tag{1.44}$$

have been introduced. It can readily be verified that

$$(\mathbf{t}_1\mathbf{d}_1) = 0, \qquad (\mathbf{t}_2\mathbf{d}_2) = 0, \tag{1.45}$$

$$(\mathbf{d}_1\mathbf{d}_2) = -(\mathbf{t}_1\mathbf{t}_2), \qquad (\mathbf{d}_1\mathbf{d}_1) = (\mathbf{t}_1\mathbf{t}_1), \qquad (\mathbf{d}_2\mathbf{d}_2) = (\mathbf{t}_2\mathbf{t}_2), \tag{1.46}$$

$$(\mathbf{d}_1\mathbf{t}_2) = (\mathbf{t}_1\mathbf{d}_2) = \sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2) - (\mathbf{t}_1\mathbf{t}_2)^2}, \tag{1.47}$$

and

$$\mathbf{t}_1\mathbf{b}_1 + \mathbf{t}_2\mathbf{b}_2 = 2|\mathbf{t}_2 \ominus \mathbf{t}_1|^2, \tag{1.48}$$

together with

$$\lim_{\mathbf{t}_2\to\mathbf{t}_1}\Big\{\mathbf{b}_1\Big\} = \lim_{\mathbf{t}_2\to\mathbf{t}_1}\Big\{\mathbf{b}_2\Big\} = 0. \tag{1.49}$$

For the products of the vectors (1.35) and (1.36) we obtain

$$(\mathbf{b}_1\mathbf{b}_1) = (\mathbf{t}_1\mathbf{t}_1) + (\mathbf{t}_2\mathbf{t}_2) - 2\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_2\mathbf{t}_2)}\,\cos\alpha + \left(\frac{1}{h^2} - 1\right)(\mathbf{t}_2\mathbf{t}_2)\sin^2\alpha,$$

$$(\mathbf{b}_2\mathbf{b}_2) = (\mathbf{t}_2\mathbf{t}_2) + (\mathbf{t}_1\mathbf{t}_1) - 2\sqrt{(\mathbf{t}_2\mathbf{t}_2)}\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\,\cos\alpha + \left(\frac{1}{h^2} - 1\right)(\mathbf{t}_1\mathbf{t}_1)\sin^2\alpha,$$

and

$$(\mathbf{b}_1\mathbf{b}_2) = -\Bigg[\left(\frac{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}}{\sqrt{(\mathbf{t}_2\mathbf{t}_2)}} + \frac{\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}} - 2\cos\alpha\right)\cos\alpha + \left(\frac{1}{h^2} - 1\right)\sin^2\alpha\Bigg](\mathbf{t}_1\mathbf{t}_2)$$

$$-\frac{1}{h}\left(\frac{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}}{\sqrt{(\mathbf{t}_2\mathbf{t}_2)}} + \frac{\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}} - 2\cos\alpha\right)\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2) - (\mathbf{t}_1\mathbf{t}_2)^2}\,\sin\alpha.$$

The following limit

$$\lim_{\mathbf{t}_2\to\mathbf{t}_1}\left\{\frac{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2)}{h\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}\frac{\sin\left[\frac{1}{h}\arccos\frac{(\mathbf{t}_1\mathbf{t}_2)}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}\right]}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2)-(\mathbf{t}_1\mathbf{t}_2)^2}}\right\}=\frac{1}{h^2} \tag{1.50}$$

is important to note.

1.2. The two-vector metric tensor and frame in quasi-euclidean space

Now we are able to introduce *the quasi-euclidean two-vector metric tensor* $n(g;\mathbf{t}_1,\mathbf{t}_2)$ by the components

$$n_{pq}(g;\mathbf{t}_1,\mathbf{t}_2)\stackrel{\text{def}}{=}\frac{\partial^2<\mathbf{t}_1,\mathbf{t}_2>}{\partial t_2^q\partial t_1^p}=-\frac{1}{2}\frac{\partial^2|\mathbf{t}_2\ominus\mathbf{t}_1|^2}{\partial t_2^q\partial t_1^p}. \tag{2.1}$$

Straightforward calculations (on the basis of (1.32) and (1.19)) show that

$$n_{pq}(g;\mathbf{t}_1,\mathbf{t}_2)=\frac{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2)}{h\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}\frac{\sin\alpha}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2)-(\mathbf{t}_1\mathbf{t}_2)^2}}r_{pq}$$

$$+\frac{1}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}A_1t_{1p}t_{2q}-\frac{1}{h\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}A_2d_{1p}d_{2q}, \tag{2.2}$$

where

$$A_1=\cos\alpha-\frac{1}{h}(\mathbf{t}_1\mathbf{t}_2)\frac{\sin\alpha}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2)-(\mathbf{t}_1\mathbf{t}_2)^2}} \tag{2.3}$$

and

$$A_2=\frac{1}{h}\cos\alpha-(\mathbf{t}_1\mathbf{t}_2)\frac{\sin\alpha}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2)-(\mathbf{t}_1\mathbf{t}_2)^2}}. \tag{2.4}$$

For the determinant of the tensor (2.2) we find simply

$$\det\left(n_{pq}(g;\mathbf{t}_1,\mathbf{t}_2)\right)=\left(\frac{\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2)}\sin\alpha}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2)-(\mathbf{t}_1\mathbf{t}_2)^2}}\right)^{N-2}h^{-N}\det\left(r_{ab}\right). \tag{2.5}$$

Owing to (1.43), we can establish the following fundamental identification:

$$\lim_{\mathbf{t}_2\to\mathbf{t}_1=\mathbf{t}}\left\{n_{pq}(g;\mathbf{t}_1,\mathbf{t}_2)\right\}=n_{pq}(g;t), \tag{2.6}$$

where $n_{pq}(g;\mathbf{t})$ is the quasi-euclidean metric tensor (see (3.49) in Part II).

Differentiating (2.2) results in

$$\frac{\partial n_{pq}(g;\mathbf{t}_1,\mathbf{t}_2)}{\partial t_1^s}=-\frac{1}{h}\frac{\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}u(\mathbf{t}_1,\mathbf{t}_2)}A_2d_{1s}r_{pq}$$

$$+\frac{1}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}A_1 t_{2q} H_{sp}(\mathbf{t}_1) + \frac{1}{h}\frac{(\mathbf{t}_1\mathbf{t}_2)}{(\mathbf{t}_1\mathbf{t}_1)\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}\frac{1}{u(\mathbf{t}_1,\mathbf{t}_2)}A_2 d_{1s} t_{1p} t_{2q}$$

$$+\frac{1}{h}\frac{1}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}\frac{1}{u(\mathbf{t}_1,\mathbf{t}_2)}A_2\times$$

$$\left[\frac{1}{(\mathbf{t}_1\mathbf{t}_1)}\left((\mathbf{t}_1\mathbf{t}_1)d_{2p}d_{2q} + (\mathbf{t}_2\mathbf{t}_2)d_{1p}d_{1q}\right)d_{1s} + (\mathbf{t}_1\mathbf{t}_2)H_{ps}(\mathbf{t}_1)d_{2q} - (\mathbf{t}_2\mathbf{t}_2)H_{qs}(\mathbf{t}_1)d_{1p}\right]$$

$$+\frac{1}{h}\frac{1}{(\mathbf{t}_1\mathbf{t}_1)\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}\left[(1-\frac{1}{h^2})\sin\alpha - \frac{(\mathbf{t}_1\mathbf{t}_2)}{u(\mathbf{t}_1,\mathbf{t}_2)}A_2\right]d_{1s}d_{1p}d_{2q}, \tag{2.7}$$

where

$$u(\mathbf{t}_1,\mathbf{t}_2) = \sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2) - (\mathbf{t}_1\mathbf{t}_2)^2} \tag{2.8}$$

and we used the relations:

$$\frac{\partial\alpha}{\partial t_1^s} = -\frac{1}{h}\frac{\mathbf{d}_1}{(\mathbf{t}_1\mathbf{t}_1)}, \qquad \frac{\partial\frac{1}{u}}{\partial t_1^s} = -\frac{1}{u^2}\mathbf{d}_2, \tag{2.9}$$

$$\frac{\partial A_1}{\partial t_1^s} = \frac{1}{h}\frac{(\mathbf{t}_1\mathbf{t}_2)}{(\mathbf{t}_1\mathbf{t}_1)u(\mathbf{t}_1,\mathbf{t}_2)}A_2 d_{1s}, \tag{2.10}$$

$$\frac{\partial A_2}{\partial t_1^s} = \frac{1}{h}\frac{(\mathbf{t}_1\mathbf{t}_2)}{(\mathbf{t}_1\mathbf{t}_1)u(\mathbf{t}_1,\mathbf{t}_2)}A_1 d_{1s} - (1-\frac{1}{h^2})\frac{(\mathbf{t}_2\mathbf{t}_2)}{u(\mathbf{t}_1,\mathbf{t}_2)}\frac{\sin\alpha}{u(\mathbf{t}_1,\mathbf{t}_2)}d_{1s}$$

$$= \frac{d_{1s}}{(\mathbf{t}_1\mathbf{t}_1)}\left[-(1-\frac{1}{h^2})\sin\alpha + \frac{(\mathbf{t}_1\mathbf{t}_2)}{u(\mathbf{t}_1,\mathbf{t}_2)}A_2\right]. \tag{2.11}$$

Since

$$\lim_{\mathbf{t}_2\to\mathbf{t}_1}\left\{A_1\right\} = 1-\frac{1}{h^2}, \qquad \lim_{\mathbf{t}_2\to\mathbf{t}_1}\left\{\frac{A_2}{u}\right\} = 0, \tag{2.12}$$

$$\lim_{\mathbf{t}_2\to\mathbf{t}_1=\mathbf{t}}\left\{\frac{\partial n_{pq}(g;\mathbf{t}_1,\mathbf{t}_2)}{\partial t_1^s}\right\} = (1-\frac{1}{h^2})\frac{t_q}{(\mathbf{t}\mathbf{t})}H_{sp}(\mathbf{t}), \tag{2.13}$$

and

$$\lim_{\mathbf{t}_2\to\mathbf{t}_1=\mathbf{t}}\left\{\frac{\partial n_{pq}(g;\mathbf{t}_1,\mathbf{t}_2)}{\partial t_2^s}\right\} = (1-\frac{1}{h^2})\frac{t_p}{(\mathbf{t}\mathbf{t})}H_{sq}(\mathbf{t}), \tag{2.14}$$

the fundamental consequence

$$\lim_{\mathbf{t}_2\to\mathbf{t}_1=\mathbf{t}}\left\{\frac{\partial n_{pq}(g;\mathbf{t}_1,\mathbf{t}_2)}{\partial t_1^s} + \frac{\partial n_{pq}(g;\mathbf{t}_1,\mathbf{t}_2)}{\partial t_2^s}\right\} = \frac{\partial n_{pq}(g;\mathbf{t})}{\partial t^s} \tag{2.15}$$

is obtained.

The expansion with respect to an appropriate orthonormal frame $f_p^R(g;\mathbf{t}_1,\mathbf{t}_2)$ can be found to read

$$n_{pq}(g;\mathbf{t}_1,\mathbf{t}_2)=\sum_{R=1}^{N} f_p^R(g;\mathbf{t}_1,\mathbf{t}_2)f_q^R(g;\mathbf{t}_2,\mathbf{t}_1) \tag{2.16}$$

with

$$\sqrt{h\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}f_p^R(g;\mathbf{t}_1,\mathbf{t}_2)=zh_p^R$$

$$-\frac{1}{(\mathbf{t}_1\mathbf{t}_2)}\Bigl[z-\sqrt{z^2+(\mathbf{t}_1\mathbf{t}_2)\Bigl(h\cos\alpha-(\mathbf{t}_1\mathbf{t}_2)\frac{\sin\alpha}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2)-(\mathbf{t}_1\mathbf{t}_2)^2}}\Bigr)}\Bigr]t_2^R t_{1p}$$

$$+\frac{1}{(\mathbf{t}_1\mathbf{t}_2)}\Bigl[z-\sqrt{z^2+(\mathbf{t}_1\mathbf{t}_2)\Bigl(\frac{1}{h}\cos\alpha-(\mathbf{t}_1\mathbf{t}_2)\frac{\sin\alpha}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2)-(\mathbf{t}_1\mathbf{t}_2)^2}}\Bigr)}\Bigr]d_{2p}d_1^R, \tag{2.17}$$

where

$$z=\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2)\frac{\sin\alpha}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2)-(\mathbf{t}_1\mathbf{t}_2)^2}}}, \tag{2.18}$$

or finally,

$$\sqrt{h\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}f_p^R(g;\mathbf{t}_1,\mathbf{t}_2)$$

$$=zh_p^R-\frac{1}{(\mathbf{t}_1\mathbf{t}_2)}\Bigl[z-\sqrt{h(\mathbf{t}_1\mathbf{t}_2)\cos\alpha+\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2)-(\mathbf{t}_1\mathbf{t}_2)^2}\sin\alpha}\Bigr]\,t_2^R t_{1p}$$

$$+\frac{1}{(\mathbf{t}_1\mathbf{t}_2)}\Bigl[z-\sqrt{\frac{1}{h}(\mathbf{t}_1\mathbf{t}_2)\cos\alpha+\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2)-(\mathbf{t}_1\mathbf{t}_2)^2}\sin\alpha}\Bigr]\,d_{2p}d_1^R. \tag{2.19}$$

Contracting the frame by vectors yields

$$f_p^R(g;\mathbf{t}_1,\mathbf{t}_2)t_1^p$$

$$=\frac{1}{\sqrt{h\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_1\mathbf{t}_2)}}}\Bigl[\frac{(\mathbf{t}_1\mathbf{t}_1)}{(\mathbf{t}_1\mathbf{t}_2)}\Bigl(\sqrt{h(\mathbf{t}_1\mathbf{t}_2)\cos\alpha+\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2)-(\mathbf{t}_1\mathbf{t}_2)^2}\ \sin\alpha}$$

$$-\sqrt{\frac{1}{h}(\mathbf{t}_1\mathbf{t}_2)\cos\alpha+\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2)-(\mathbf{t}_1\mathbf{t}_2)^2}\,\sin\alpha}\Biggr)\,t_2^R$$

$$+\sqrt{\frac{1}{h}(\mathbf{t}_1\mathbf{t}_2)\cos\alpha+\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2)-(\mathbf{t}_1\mathbf{t}_2)^2}\,\sin\alpha}\,t_1^R\Bigr], \tag{2.20}$$

$$f_p^R(g;\mathbf{t}_1,\mathbf{t}_2)t_2^p$$

$$=\frac{1}{\sqrt{h\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}}\sqrt{h(\mathbf{t}_1\mathbf{t}_2)\cos\alpha+\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2)-(\mathbf{t}_1\mathbf{t}_2)^2}\,\sin\alpha}\,t_2^R, \tag{2.21}$$

and

$$\sum_{R=1}^{N} f_p^R(g;\mathbf{t}_1,\mathbf{t}_2)t_1^R=$$

$$=\frac{1}{\sqrt{h\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}}\sqrt{h(\mathbf{t}_1\mathbf{t}_2)\cos\alpha+\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2)-(\mathbf{t}_1\mathbf{t}_2)^2}\,\sin\alpha}\;t_{1p}, \tag{2.22}$$

together with

$$\sum_{R=1}^{N} f_p^R(g;\mathbf{t}_1,\mathbf{t}_2)t_2^R=$$

$$=\frac{1}{\sqrt{h\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_1\mathbf{t}_2)}}}\Bigl[\frac{(\mathbf{t}_2\mathbf{t}_2)}{(\mathbf{t}_1\mathbf{t}_2)}\Biggl(\sqrt{h(\mathbf{t}_1\mathbf{t}_2)\cos\alpha+\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2)-(\mathbf{t}_1\mathbf{t}_2)^2}\,\sin\alpha}$$

$$-\sqrt{\frac{1}{h}(\mathbf{t}_1\mathbf{t}_2)\cos\alpha+\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2)-(\mathbf{t}_1\mathbf{t}_2)^2}\,\sin\alpha}\Biggr)\,t_{1p}$$

$$+\sqrt{\frac{1}{h}(\mathbf{t}_1\mathbf{t}_2)\cos\alpha+\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2)-(\mathbf{t}_1\mathbf{t}_2)^2}\,\sin\alpha}\;t_{2p}\Bigr]. \tag{2.23}$$

1.3. Covariant version

It proves possible to convert the approach into *the co-version* by introducing *the co-vectors*

$$T_{1p}(g;\mathbf{t}_1,\mathbf{t}_2)\overset{\text{def}}{=}n_{pq}(g;\mathbf{t}_1,\mathbf{t}_2)\mathbf{t}_2^q,\qquad T_{2q}(g;\mathbf{t}_1,\mathbf{t}_2)\overset{\text{def}}{=}\mathbf{t}_1^p n_{pq}(g;\mathbf{t}_1,\mathbf{t}_2). \tag{3.1}$$

Using (2.2), we get

$$\mathbf{T}_1 = \frac{\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}}\mathbf{t}_1 \cos\alpha + \frac{\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}{h\sqrt{(\mathbf{t}_1\mathbf{t}_1)}}\mathbf{d}_1 \sin\alpha \tag{3.2}$$

and

$$\mathbf{T}_2 = \frac{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}}{\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}\mathbf{t}_2 \cos\alpha + \frac{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}}{h\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}\mathbf{d}_2 \sin\alpha. \tag{3.3}$$

The equality

$$\mathbf{t}_1\mathbf{T}_1 + \mathbf{t}_2\mathbf{T}_2 = 2 < \mathbf{t}_1, \mathbf{t}_2 >= 2\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_2\mathbf{t}_2)}\cos\alpha \tag{3.4}$$

holds. Also,

$$\lim_{\mathbf{t}_2\to\mathbf{t}_1=\mathbf{t}}\Big\{\mathbf{T}_1\Big\} = \lim_{\mathbf{t}_2\to\mathbf{t}_1=\mathbf{t}}\Big\{\mathbf{T}_2\Big\} = \mathbf{t}. \tag{3.5}$$

The metric tensor (2.1)-(2.2) is obtainable from these vectors as follows:

$$n_{pq}(g;\mathbf{t}_1,\mathbf{t}_2) = \frac{\partial T_{1p}}{\partial t_2^q} = \frac{\partial T_{2q}}{\partial t_1^p}. \tag{3.6}$$

The respective products are found to be

$$(\mathbf{T}_1\mathbf{T}_1) = (\mathbf{t}_2\mathbf{t}_2)(\cos^2\alpha + \frac{1}{h^2}\sin^2\alpha), \qquad (\mathbf{T}_2\mathbf{T}_2) = (\mathbf{t}_1\mathbf{t}_1)(\cos^2\alpha + \frac{1}{h^2}\sin^2\alpha), \tag{3.7}$$

and

$$(\mathbf{T}_1\mathbf{T}_2) = (\cos^2\alpha - \frac{1}{h^2}\sin^2\alpha)(\mathbf{t}_1\mathbf{t}_2) + 2\frac{1}{h}\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2) - (\mathbf{t}_1\mathbf{t}_2)^2}\,\cos\alpha\sin\alpha, \tag{3.8}$$

together with

$$\sqrt{(\mathbf{T}_1\mathbf{T}_1)(\mathbf{T}_2\mathbf{T}_2) - (\mathbf{T}_1\mathbf{T}_2)^2}$$

$$= \frac{2}{h}(\mathbf{t}_1\mathbf{t}_2)\sin\alpha\cos\alpha - (\cos^2\alpha - \frac{1}{h^2}\sin^2\alpha)\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2) - (\mathbf{t}_1\mathbf{t}_2)^2}. \tag{3.9}$$

From (3.7)-(3.8) it follows that

$$(\cos^2\alpha + \frac{1}{h^2}\sin^2\alpha)^2\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2) - (\mathbf{t}_1\mathbf{t}_2)^2}$$

$$= \frac{2}{h}(\mathbf{T}_1\mathbf{T}_2)\sin\alpha\cos\alpha - (\cos^2\alpha - \frac{1}{h^2}\sin^2\alpha)\sqrt{(\mathbf{T}_1\mathbf{T}_1)(\mathbf{T}_2\mathbf{T}_2) - (\mathbf{T}_1\mathbf{T}_2)^2} \tag{3.10}$$

and

$$(\cos^2\alpha + \frac{1}{h^2}\sin^2\alpha)^2(\mathbf{t}_1\mathbf{t}_2)$$

$$= (\cos^2\alpha - \frac{1}{h^2}\sin^2\alpha)(\mathbf{T}_1\mathbf{T}_2) + \frac{2}{h}\sin\alpha\cos\alpha\sqrt{(\mathbf{T}_1\mathbf{T}_1)(\mathbf{T}_2\mathbf{T}_2) - (\mathbf{T}_1\mathbf{T}_2)^2}, \qquad (3.11)$$

together with

$$(\cos^2\alpha + \frac{1}{h^2}\sin^2\alpha)\Big[-\frac{1}{h}(\mathbf{t}_1\mathbf{t}_2)\sin\alpha + \sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2) - (\mathbf{t}_1\mathbf{t}_2)^2}\ \cos\alpha\Big]$$

$$= \frac{1}{h}(\mathbf{T}_1\mathbf{T}_2)\sin\alpha - \sqrt{(\mathbf{T}_1\mathbf{T}_1)(\mathbf{T}_2\mathbf{T}_2) - (\mathbf{T}_1\mathbf{T}_2)^2}\ \cos\alpha. \qquad (3.12)$$

Using these formulas in calculating the co-representation

$$\alpha = \alpha(h; \mathbf{T}_1, \mathbf{T}_2) \qquad (3.13)$$

for the angle (1.18) yields the following implicit equation:

$$\cos(h\alpha) = \frac{(\cos^2\alpha - \frac{1}{h^2}\sin^2\alpha)(\mathbf{T}_1\mathbf{T}_2) + \frac{2}{h}\sin\alpha\cos\alpha\sqrt{(\mathbf{T}_1\mathbf{T}_1)(\mathbf{T}_2\mathbf{T}_2) - (\mathbf{T}_1\mathbf{T}_2)^2}}{(\cos^2\alpha + \frac{1}{h^2}\sin^2\alpha)\sqrt{(\mathbf{T}_1\mathbf{T}_1)}\sqrt{(\mathbf{T}_2\mathbf{T}_2)}}. \qquad (3.14)$$

The respective co-version of the scalar product (1.30) reads

$$< \mathbf{T}_2, \mathbf{T}_2 >= \sqrt{(\mathbf{T}_1\mathbf{T}_1)}\sqrt{(\mathbf{T}_2\mathbf{T}_2)}\cos\alpha. \qquad (3.15)$$

On this way the set (3.2)-(3.3) can be inverted, yielding

$$\mathbf{t}_1(g; \mathbf{T}_1, \mathbf{T}_2) = \frac{1}{f}\Big[\frac{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}}{\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}\left(\cos\alpha - \frac{1}{h}\frac{(\mathbf{t}_1\mathbf{t}_2)}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2) - (\mathbf{t}_1\mathbf{t}_2)^2}}\sin\alpha\right)\mathbf{T}_1$$

$$- \frac{1}{h}\frac{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2) - (\mathbf{t}_1\mathbf{t}_2)^2}}\sin\alpha\ \mathbf{T}_2\Big] \qquad (3.16)$$

and

$$\mathbf{t}_2(g; \mathbf{T}_1, \mathbf{T}_2) = \frac{1}{f}\Big[\frac{\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}}\left(\cos\alpha - \frac{1}{h}\frac{(\mathbf{t}_1\mathbf{t}_2)}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2) - (\mathbf{t}_1\mathbf{t}_2)^2}}\sin\alpha\right)\mathbf{T}_2$$

$$- \frac{1}{h}\frac{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_2\mathbf{t}_2)}}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2) - (\mathbf{t}_1\mathbf{t}_2)^2}}\sin\alpha\ \mathbf{T}_1\Big], \qquad (3.17)$$

where

$$f = \left(\cos\alpha - \frac{1}{h}\frac{(\mathbf{t}_1\mathbf{t}_2)}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2) - (\mathbf{t}_1\mathbf{t}_2)^2}}\sin\alpha\right)^2$$

$$-\frac{1}{h^2}\left(\frac{\sin\alpha}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2)-(\mathbf{t}_1\mathbf{t}_2)^2}}\right)^2(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2), \tag{3.18}$$

or

$$f=\cos^2\alpha-\frac{1}{h^2}\sin^2\alpha-\frac{2}{h}\frac{\sin\alpha\cos\alpha}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2)-(\mathbf{t}_1\mathbf{t}_2)^2}}(\mathbf{t}_1\mathbf{t}_2). \tag{3.19}$$

Taking into account (3.9), this function can be written merely as

$$f=-\frac{\sqrt{(\mathbf{T}_1\mathbf{T}_1)(\mathbf{T}_2\mathbf{T}_2)-(\mathbf{T}_1\mathbf{T}_2)^2}}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)(\mathbf{t}_2\mathbf{t}_2)-(\mathbf{t}_1\mathbf{t}_2)^2}}. \tag{3.20}$$

Thus we find

$$\mathbf{t}_1=\frac{\sqrt{(\mathbf{T}_2\mathbf{T}_2)}}{\sqrt{(\mathbf{T}_1\mathbf{T}_1)}}\mathbf{T}_1\frac{\cos\alpha}{\cos^2\alpha+\dfrac{1}{h^2}\sin^2\alpha}+\frac{1}{h}\frac{\sqrt{(\mathbf{T}_2\mathbf{T}_2)}}{\sqrt{(\mathbf{T}_1\mathbf{T}_1)}}\mathbf{D}_1\frac{\sin\alpha}{\cos^2\alpha+\dfrac{1}{h^2}\sin^2\alpha} \tag{3.21}$$

and

$$\mathbf{t}_2=\frac{\sqrt{(\mathbf{T}_1\mathbf{T}_1)}}{\sqrt{(\mathbf{T}_2\mathbf{T}_2)}}\mathbf{T}_2\frac{\cos\alpha}{\cos^2\alpha+\dfrac{1}{h^2}\sin^2\alpha}+\frac{1}{h}\frac{\sqrt{(\mathbf{T}_1\mathbf{T}_1)}}{\sqrt{(\mathbf{T}_2\mathbf{T}_2)}}\mathbf{D}_2\frac{\sin\alpha}{\cos^2\alpha+\dfrac{1}{h^2}\sin^2\alpha}, \tag{3.22}$$

where

$$\mathbf{D}_1=\frac{(\mathbf{T}_1\mathbf{T}_1)\mathbf{T}_2-(\mathbf{T}_1\mathbf{T}_2)\mathbf{T}_1}{\sqrt{(\mathbf{T}_1\mathbf{T}_1)(\mathbf{T}_2\mathbf{T}_2)-(\mathbf{T}_1\mathbf{T}_2)^2}}, \qquad \mathbf{D}_2=\frac{(\mathbf{T}_2\mathbf{T}_2)\mathbf{T}_1-(\mathbf{T}_1\mathbf{T}_2)\mathbf{T}_2}{\sqrt{(\mathbf{T}_1\mathbf{T}_1)(\mathbf{T}_2\mathbf{T}_2)-(\mathbf{T}_1\mathbf{T}_2)^2}}. \tag{3.23}$$

Similarly to (1.38)-(1.40), the identities

$$(\mathbf{T}_1\mathbf{D}_1)=0, \qquad (\mathbf{T}_2\mathbf{D}_2)=0, \tag{3.24}$$

$$(\mathbf{D}_1\mathbf{D}_2)=-(\mathbf{T}_1\mathbf{T}_2), \qquad (\mathbf{D}_1\mathbf{D}_1)=(\mathbf{T}_1\mathbf{T}_1), \qquad (\mathbf{D}_2\mathbf{D}_2)=(\mathbf{T}_2\mathbf{T}_2), \tag{3.25}$$

and

$$(\mathbf{D}_1\mathbf{T}_2)=(\mathbf{T}_1\mathbf{D}_2)=\sqrt{(\mathbf{T}_1\mathbf{T}_1)(\mathbf{T}_2\mathbf{T}_2)-(\mathbf{T}_1\mathbf{T}_2)^2} \tag{3.26}$$

hold.

By the help of (3.20)-(3.21), and in close similarity to (3.6), the co-version

$$N^{pq}(g;\mathbf{T}_1,\mathbf{T}_2)\stackrel{\text{def}}{=}\frac{\partial t_1^p}{\partial T_{2q}}=\frac{\partial t_2^q}{\partial T_{1p}} \tag{3.27}$$

of the two-vector metric tensor (2.2) can be arrived at.

1.4. $\mathcal{E}_g^{PD}$-parallelogram law

Let $\mathbf{t}_1, \mathbf{t}_2$, and $\mathbf{t}_3$ be three vectors issued from the same origin O, subject to the conditions that the angle between $\mathbf{t}_1$ and $\mathbf{t}_2$ is acute and the vector $\mathbf{t}_3$ is positioned

between the vectors $\mathbf{t}_1$ and $\mathbf{t}_2$. Let us denote the end points of the vectors $\mathbf{t}_1, \mathbf{t}_2$, and $\mathbf{t}_3$ as X_1, X_2, and X_3, respectively. On joining the points X_1 and X_3, and also X_2 and X_3, by $\mathcal{E}_g^{PD}$-geodesics, we get a tetragonal figure, to be denoted as $\mathcal{P}_4$.

Using Eqs. (1.14), (1.15), and (1.27), we can set forth the following couple of two equations:

$$(\mathbf{t}_2\mathbf{t}_2) = (\mathbf{t}_1\mathbf{t}_1) + (\mathbf{t}_3\mathbf{t}_3) - 2\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\,\sqrt{(\mathbf{t}_3\mathbf{t}_3)}\cos\Big[\frac{1}{h}\arccos\frac{(\mathbf{t}_1\mathbf{t}_3)}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\,\sqrt{(\mathbf{t}_3\mathbf{t}_3)}}\Big] \tag{4.1}$$

and

$$(\mathbf{t}_1\mathbf{t}_1) = (\mathbf{t}_3\mathbf{t}_3) + (\mathbf{t}_2\mathbf{t}_2) - 2\sqrt{(\mathbf{t}_3\mathbf{t}_3)}\,\sqrt{(\mathbf{t}_2\mathbf{t}_2)}\cos\Big[\frac{1}{h}\arccos\frac{(\mathbf{t}_2\mathbf{t}_3)}{\sqrt{(\mathbf{t}_2\mathbf{t}_2)}\,\sqrt{(\mathbf{t}_3\mathbf{t}_3)}}\Big], \tag{4.2}$$

which can also be rewritten in the convenient form

$$\sqrt{(\mathbf{t}_3\mathbf{t}_3)} - \frac{(\mathbf{t}_2\mathbf{t}_2) - (\mathbf{t}_1\mathbf{t}_1)}{\sqrt{(\mathbf{t}_3\mathbf{t}_3)}} = 2\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\cos\Big[\frac{1}{h}\arccos\frac{(\mathbf{t}_1\mathbf{t}_3)}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\,\sqrt{(\mathbf{t}_3\mathbf{t}_3)}}\Big] \tag{4.3}$$

and

$$\sqrt{(\mathbf{t}_3\mathbf{t}_3)} - \frac{(\mathbf{t}_1\mathbf{t}_1) - (\mathbf{t}_2\mathbf{t}_2)}{\sqrt{(\mathbf{t}_3\mathbf{t}_3)}} = 2\sqrt{(\mathbf{t}_2\mathbf{t}_2)}\cos\Big[\frac{1}{h}\arccos\frac{(\mathbf{t}_2\mathbf{t}_3)}{\sqrt{(\mathbf{t}_2\mathbf{t}_2)}\,\sqrt{(\mathbf{t}_3\mathbf{t}_3)}}\Big]. \tag{4.4}$$

In (4.1), the left-hand part is the squared length of the straight side OX_2 and the right-hand side is the squared length of the geodesic side X_1X_3. According to (4.2), the lengths of OX_1 and X_2X_3 are equal. Under these conditions, the figure $\mathcal{P}_4$ does attribute the general property of the euclidean parallelogram that the lengths of opposite sides are equal. In this vein, we introduce the following

DEFINITION. Subject to the equations (4.1) and (4.2), the tetragonal figure $\mathcal{P}_4$ is called *the $\mathcal{E}_g^{PD}$-parallelogram,* and the vector $\mathbf{t}_3$ is called the $\mathcal{E}_g^{PD}$-sum vector:

$$\mathbf{t}_3 = \mathbf{t}_1 \oplus \mathbf{t}_2. \tag{4.5}$$

NOTE. The qualitative distinction here from euclidean patterns is that the sides X_1X_3 and X_3X_2 of the $\mathcal{P}_4$ are curved lines in general, namely geodesic arcs, which generally cease to be straight under the $\mathcal{E}_g^{PD}$-extension.

Finding the sum vector (4.5) implies solving the set of the equations (4.3) and (4.4). It proves easy to proceed approximately, namely taking

$$\frac{1}{h} = 1 + k \tag{4.6}$$

and

$$\mathbf{t}_3 = \mathbf{t}_1 + \mathbf{t}_2 + k\mathbf{c}(\mathbf{t}_1, \mathbf{t}_2), \quad k \ll 1. \tag{4.7}$$

Under these conditions, on inserting (4.6) and (4.7) in (4.3), we find

$$\sqrt{(\mathbf{t}_2 + \mathbf{t}_1)^2} + k\frac{(\mathbf{t}_2 + \mathbf{t}_1)\mathbf{c}}{\sqrt{(\mathbf{t}_2 + \mathbf{t}_1)^2}} - \frac{(\mathbf{t}_2\mathbf{t}_2) - (\mathbf{t}_1\mathbf{t}_1)}{\sqrt{(\mathbf{t}_2 + \mathbf{t}_1)^2}}\left(1 - k\frac{(\mathbf{t}_2 + \mathbf{t}_1)\mathbf{c}}{(\mathbf{t}_2 + \mathbf{t}_1)^2}\right)$$

$$= 2\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\cos\Bigl[(1+k)\arccos\frac{(\mathbf{t}_1\mathbf{t}_3)}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_3\mathbf{t}_3)}}\Bigr]$$

$$= 2\frac{(\mathbf{t}_1\mathbf{t}_3)}{\sqrt{(\mathbf{t}_3\mathbf{t}_3)}} - 2k\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{1-\left(\frac{\mathbf{t}_1(\mathbf{t}_2+\mathbf{t}_1)}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_2+\mathbf{t}_1)^2}}\right)^2}\arccos\frac{\mathbf{t}_1(\mathbf{t}_2+\mathbf{t}_1)}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_2+\mathbf{t}_1)^2}}$$

$$= 2k\frac{(\mathbf{t}_1\mathbf{c})}{\sqrt{(\mathbf{t}_2+\mathbf{t}_1)^2}} + 2\frac{\mathbf{t}_1(\mathbf{t}_2+\mathbf{t}_1)}{\sqrt{(\mathbf{t}_2+\mathbf{t}_1)^2}}\left(1-k\frac{(\mathbf{t}_2+\mathbf{t}_1)\mathbf{c}}{(\mathbf{t}_2+\mathbf{t}_1)^2}\right)$$

$$- 2k\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{1-\left(\frac{\mathbf{t}_1(\mathbf{t}_2+\mathbf{t}_1)}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_2+\mathbf{t}_1)^2}}\right)^2}\arccos\frac{\mathbf{t}_1(\mathbf{t}_2+\mathbf{t}_1)}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_2+\mathbf{t}_1)^2}}, \tag{4.8}$$

which entails

$$\frac{(\mathbf{t}_2+\mathbf{t}_1)\mathbf{c}}{\sqrt{(\mathbf{t}_2+\mathbf{t}_1)^2}} + \frac{(\mathbf{t}_2\mathbf{t}_2)-(\mathbf{t}_1\mathbf{t}_1)}{(\mathbf{t}_2+\mathbf{t}_1)^2}\frac{(\mathbf{t}_2+\mathbf{t}_1)\mathbf{c}}{\sqrt{(\mathbf{t}_2+\mathbf{t}_1)^2}}$$

$$= 2\frac{(\mathbf{t}_1\mathbf{c})}{\sqrt{(\mathbf{t}_2+\mathbf{t}_1)^2}} - 2\frac{\mathbf{t}_1(\mathbf{t}_2+\mathbf{t}_1)}{(\mathbf{t}_2+\mathbf{t}_1)^2}\frac{(\mathbf{t}_2+\mathbf{t}_1)\mathbf{c}}{\sqrt{(\mathbf{t}_2+\mathbf{t}_1)^2}}$$

$$- 2\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{1-\left(\frac{\mathbf{t}_1(\mathbf{t}_2+\mathbf{t}_1)}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_2+\mathbf{t}_1)^2}}\right)^2}\arccos\frac{\mathbf{t}_1(\mathbf{t}_2+\mathbf{t}_1)}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_2+\mathbf{t}_1)^2}}. \tag{4.9}$$

Therefore we obtain

$$\mathbf{t}_2\mathbf{c} = -u(\mathbf{t}_1,\mathbf{t}_2)\arccos\frac{\mathbf{t}_1(\mathbf{t}_2+\mathbf{t}_1)}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_2+\mathbf{t}_1)^2}}. \tag{4.10}$$

Similarly, from (4.4) it follows that

$$\mathbf{t}_1\mathbf{c} = -u(\mathbf{t}_1,\mathbf{t}_2)\arccos\frac{\mathbf{t}_2(\mathbf{t}_2+\mathbf{t}_1)}{\sqrt{(\mathbf{t}_2\mathbf{t}_2)}\sqrt{(\mathbf{t}_2+\mathbf{t}_1)^2}}, \tag{4.11}$$

where $u(\mathbf{t}_1,\mathbf{t}_2)$ is the function (2.8).

If we use now the symmetrized expansion

$$\mathbf{c} = m(\mathbf{t}_1,\mathbf{t}_2)\mathbf{t}_1 + n(\mathbf{t}_1,\mathbf{t}_2)\mathbf{t}_2, \tag{4.12}$$

then we find

$$m(\mathbf{t}_1,\mathbf{t}_2) =$$

$$\frac{1}{u(\mathbf{t}_1,\mathbf{t}_2)}\left((\mathbf{t}_1\mathbf{t}_2)\arccos\frac{\mathbf{t}_1(\mathbf{t}_2+\mathbf{t}_1)}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_2+\mathbf{t}_1)^2}}-(\mathbf{t}_2\mathbf{t}_2)\arccos\frac{\mathbf{t}_2(\mathbf{t}_2+\mathbf{t}_1)}{\sqrt{(\mathbf{t}_2\mathbf{t}_2)}\sqrt{(\mathbf{t}_2+\mathbf{t}_1)^2}}\right) \tag{4.13}$$

and

$$n(\mathbf{t}_1,\mathbf{t}_2)=$$

$$\frac{1}{u(\mathbf{t}_1,\mathbf{t}_2)}\left((\mathbf{t}_1\mathbf{t}_2)\arccos\frac{\mathbf{t}_2(\mathbf{t}_2+\mathbf{t}_1)}{\sqrt{(\mathbf{t}_2\mathbf{t}_2)}\sqrt{(\mathbf{t}_2+\mathbf{t}_1)^2}}-(\mathbf{t}_1\mathbf{t}_1)\arccos\frac{\mathbf{t}_1(\mathbf{t}_2+\mathbf{t}_1)}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_2+\mathbf{t}_1)^2}}\right). \tag{4.14}$$

Since

$$m(\mathbf{t}_1,\mathbf{t}_2)=n(\mathbf{t}_2,\mathbf{t}_1), \tag{4.15}$$

we just deduce the approximate solution

$$\mathbf{t}_1\oplus\mathbf{t}_2\approx\mathbf{t}_1+\mathbf{t}_2+(\frac{1}{h}-1)\Big(m(\mathbf{t}_1,\mathbf{t}_2)\mathbf{t}_1+m(\mathbf{t}_2,\mathbf{t}_1)\mathbf{t}_2\Big),\qquad \frac{1}{h}-1\ll 1. \tag{4.16}$$

Alternatively, the solution $\mathbf{t}_2=\mathbf{t}_2(\mathbf{t}_1,\mathbf{t}_3)$ to the set of equations (4.1)-(4.2) can naturally be treated as *the* $\mathcal{E}_g^{PD}$*-difference* of vectors $\mathbf{t}_3$ and $\mathbf{t}_1$:

$$\mathbf{t}_2=\mathbf{t}_3\ominus\mathbf{t}_1. \tag{4.17}$$

Again, restricting ourselves to the approximation, from (4.1)-(4.2) we obtain

$$\mathbf{t}_3\ominus\mathbf{t}_1\approx\mathbf{t}_3-\mathbf{t}_1+(\frac{1}{h}-1)\mathbf{s}(\mathbf{t}_1,\mathbf{t}_3),\qquad \frac{1}{h}-1\ll 1, \tag{4.18}$$

with

$$\mathbf{s}(\mathbf{t}_1,\mathbf{t}_3)=\frac{1}{u(\mathbf{t}_1,\mathbf{t}_3)}\Big\{\Big[(\mathbf{t}_1\mathbf{t}_1)\arccos\frac{(\mathbf{t}_1\mathbf{t}_3)}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_3\mathbf{t}_3)}}$$

$$-(\mathbf{t}_3-\mathbf{t}_1,\mathbf{t}_1)\arccos\frac{(\mathbf{t}_3-\mathbf{t}_1,\mathbf{t}_3)}{\sqrt{(\mathbf{t}_3-\mathbf{t}_1,\mathbf{t}_3-\mathbf{t}_1)}\sqrt{(\mathbf{t}_3\mathbf{t}_3)}}\Big](\mathbf{t}_3-\mathbf{t}_1)$$

$$+\Big[(\mathbf{t}_3-\mathbf{t}_1,\mathbf{t}_3-\mathbf{t}_1)\arccos\frac{(\mathbf{t}_3-\mathbf{t}_1,\mathbf{t}_3)}{\sqrt{(\mathbf{t}_3-\mathbf{t}_1,\mathbf{t}_3-\mathbf{t}_1)}\sqrt{(\mathbf{t}_3\mathbf{t}_3)}}$$

$$-(\mathbf{t}_3-\mathbf{t}_1,\mathbf{t}_1)\arccos\frac{(\mathbf{t}_1\mathbf{t}_3)}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_3\mathbf{t}_3)}}\Big]\mathbf{t}_1\Big\}. \tag{4.19}$$

Here it is useful to note that

$$(\mathbf{t}_3-\mathbf{t}_1,\mathbf{s})=u(\mathbf{t}_1,\mathbf{t}_3)\arccos\frac{(\mathbf{t}_1\mathbf{t}_3)}{\sqrt{(\mathbf{t}_1\mathbf{t}_1)}\sqrt{(\mathbf{t}_3\mathbf{t}_3)}}, \tag{4.20}$$

$$(\mathbf{t}_1,\mathbf{s})=u(\mathbf{t}_1,\mathbf{t}_3)\arccos\frac{(\mathbf{t}_3-\mathbf{t}_1,\mathbf{t}_3)}{\sqrt{(\mathbf{t}_3-\mathbf{t}_1,\mathbf{t}_3-\mathbf{t}_1)}\sqrt{(\mathbf{t}_3\mathbf{t}_3)}}, \tag{4.21}$$

and

$$u(\mathbf{t}_3 - \mathbf{t}_1, \mathbf{t}_3) = u(\mathbf{t}_1, \mathbf{t}_3). \tag{4.22}$$

The problem of finding the sum vector $\mathbf{t}_1 \oplus \mathbf{t}_2$ and the difference vector $\mathbf{t}_3 \ominus \mathbf{t}_2$ in general exact forms is open and seems to be difficult.

1.5. Return to initial $\mathcal{E}_g^{PD}$-space

Applying the quasi-euclidean transformation (see (3.11) in Part II) to (1.28) results in the following $\mathcal{E}_g^{PD}$*-scalar product*:

$$< R, S >= K(g;R)K(g;S)\cos\Big[\frac{1}{h}\arccos\frac{A(g;R)A(g;S) + h^2 r_{be}R^b S^e}{\sqrt{B(g;R)}\sqrt{B(g;S)}}\Big], \tag{5.1}$$

so that *the* $\mathcal{E}_g^{PD}$*-angle*

$$\alpha = \frac{1}{h}\arccos\frac{A(g;R)A(g;S) + h^2 r_{be}R^b S^e}{\sqrt{B(g;R)}\sqrt{B(g;S)}} \tag{5.2}$$

is appeared; the functions B, K and A can be found in Sec. 2 of Part II.

Differentiating (5.1) yields

$$\frac{\partial < R, S >}{\partial R^p} = R_p \frac{< R, S >}{K^2(g;R)} + hK(g;S)s_p(g;R,S)\sin\alpha \tag{5.3}$$

and

$$\frac{\partial < R, S >}{\partial S^q} = S_q \frac{< R, S >}{K^2(g;S)} + hK(g;R)s_p(g;S,R)\sin\alpha. \tag{5.4}$$

For the associated $\mathcal{E}_g^{PD}$*-two-vector metric tensor*

$$G_{pq}(g;R,S) \stackrel{\text{def}}{=} \frac{\partial^2 < R, S >}{\partial S^q \partial R^p} \tag{5.5}$$

we can find explicitly the representation

$$G_{pq}(g;R,S) = \left(\frac{R_p}{K(g;R)}\frac{S_q}{K(g;S)} - h^2 s_p(g;R,S)s_q(g;S,R)\right)\cos\alpha$$

$$+ h\Big[\left(\frac{R_p}{K(g;R)}s_q(g;S,R) + \frac{S_q}{K(g;S)}s_p(g;R,S)\right) + s_{pq}(g;R,S)\Big]\sin\alpha, \tag{5.6}$$

where

$$s_p(g;R,S)=\frac{M_p(g;R,S)}{W(g;R,S)}\frac{K(g;R)}{B(g;R)}$$

and

$$s_{pq}(g;R,S)=K(g;S)\frac{\partial s_p(g;R,S)}{\partial S^q}$$

with

$$W(g;R,S)=\sqrt{B(g;R)B(g;S)-\left[A(g;R)A(g;S)+h^2r_{be}R^bS^e\right]^2}$$

and

$$M_p(g;R,S)=B(g;R)\sqrt{B(g;R)}\,\sqrt{B(g;S)}\frac{1}{h^2}\frac{\partial}{\partial R^p}\frac{A(g;R)A(g;S)+h^2r_{be}R^bS^e}{\sqrt{B(g;R)}\,\sqrt{B(g;S)}}.$$

The latter vector has the components

$$h^2M_N(g;R,S)=B(g;R)A(g;S)-\left[A(g;R)A(g;S)+h^2r_{be}R^bS^e\right]A(g;R)$$

and

$$h^2M_a(g;R,S)=B(g;R)\left(\frac{1}{2}g\frac{R^b}{q(R)}A(g;S)+h^2S^b\right)r_{ab}$$

$$-\left[A(g;R)A(g;S)+h^2r_{be}R^bS^e\right]\left(\frac{1}{2}gR^N+q(R)\right)\frac{R^b}{q(R)}r_{ab},$$

which can be simplified to get

$$M_N(g;R,S)=q^2(R)A(g;S)-r_{be}R^bS^eA(g;R)$$

and

$$M_a(g;R,S)=\left(-R^NR^bA(g;S)+S^bB(g;R)-r_{ec}R^eS^c\left(q(R)+\frac{1}{2}gR^N\right)\frac{R^b}{q(R)}\right)r_{ab}.$$

The identity

$$M_p(g;R,S)R^p=0 \tag{5.7}$$

holds.

The symmetry

$$G_{pq}(g;R,S)=G_{qp}(g;S,R) \tag{5.8}$$

and the Finslerian limit

$$\lim_{S^p\to R^p}\left\{G_{pq}(g;R,S)\right\}=g_{pq}(g;R) \tag{5.9}$$

can straightforwardly be verified; the components $g_{pq}(g;R)$ are presented in Part II by the list (2.60)-(2.61). The $\mathcal{E}_g^{PD}$-metric tensor (5.6) can also be obtained as the transform

$$G_{pq}(g;R,S)=\sigma_p^r(g;R)\sigma_q^s(g;S)n_{rs}(g;\mathbf{t}_1,\mathbf{t}_2) \tag{5.10}$$

(cf. Eq. (3.47) in Part II) of the two-vector quasi-euclidean tensor (2.2), where

$$t_1^r=\sigma^r(g;R), \qquad t_2^s=\sigma^s(g;S)$$

(cf. Eqs. (3.10) and (3.35) in Part II).

At equal vectors the two-vector scalar product (5.1) is exactly the squared Finslerian metric function:

$$<R,R>=K^2(g;R), \tag{5.11}$$

where $K(g;R)$ is the function given in Part II by Eq. (2.30).

In the original $\mathcal{E}_g^{PD}$-space, the general solution to the geodesic equation (1.1) reads

$$R^p(s)=\mu^p(g;t(g;s)), \tag{5.12}$$

where $t(g;s)$ is given by (1.35), and μ^p are the functions which realize the quasi-euclidean transformation according to Eqs. (3.14)-(3.15) of Part II.

Particularly, from (5.2) it directly ensues that the angle value α of a vector R with the R^N-axis is equal to

$$\alpha=\frac{1}{h}\arccos\frac{A(g;R)}{\sqrt{B(g;R)}} \tag{5.13}$$

and with respect to the $\{\mathbf{R}\}$-plane is equal to

$$\alpha=\frac{1}{h}\arccos\frac{L(g;R)}{\sqrt{B(g;R)}}; \tag{5.14}$$

here, B, L, and A are respectively the functions (2.30), (2.36), and (2.39) of Part II.

PART II:

FINSLEROID–SPACE $\mathcal{E}_g^{PD}$ OF POSITIVE–DEFINITE TYPE

2.1. Motivation

Below Sec. 2.2 presents the notation and the conventions for the space $\mathcal{E}_g^{PD}$ and introduces necessary initial concepts and definitions. The space $\mathcal{E}_g^{PD}$ is constructed by assuming an axial symmetry and, therefore, the space incorporates a single preferred direction, which we shall often refer as the Z-axis. The abbreviations FMF and FMT will be used for the Finslerian metric function and the Finslerian metric tensor, respectively.

A characteristic parameter g may take on the values between -2 and 2; at $g=0$ the space $\mathcal{E}_g^{PD}$ is reduced to become an ordinary euclidean one. After preliminary introducing a characteristic quadratic form B, which is distinct from the euclidean sum of squares by occurrence of a mixed term (see Eq. (2.22)), we define the FMF K for the space $\mathcal{E}_g^{PD}$ by the help of the formulae (2.30)-(2.33). A characteristic feature of the formulae is the occurence of the function "arctan". Next, we calculate basic tensor quantities

of the space. There appears a remarkable phenomenon, which essentially simplify all the constructions, that the associated Cartan tensor proves to be of a simple algebraic structure (see Eqs. (2.66)-(2.67)). In particular, the phenomenon gives rise to a conclusion that the indicatrix (the extension of the sphere) of the space $\mathcal{E}_g^{PD}$ is a space of constant curvature. The value of the curvature depends on the parameter g according to the law (2.73).

Sec. 2.3 introduces the idea of quasi-euclidean map for the $\mathcal{E}_g^{PD}$-space. The idea is fruitful in that the quasi-euclidean space is simple in many aspects, so that the relevant transformation makes reduce various calculations. Last Sec. 2.4 offers nearest interesting properties of the quasi-euclidean metric tensor.

2.2. Initial items

Suppose we are given an N–dimensional vector space V_N. Denote by R the vectors constituting the space, so that $R \in V_N$. Any given vector R assigns a particular direction in V_N. Let us fix a member $R_{(N)} \in V_N$, introduce the straightline e_N oriented along the vector $R_{(N)}$, and use this e_N to serve as a R^N–coordinate axis in V_N. In this way we get the topological product

$$V_N = V_{N-1} \times e_N \tag{2.1}$$

together with the separation

$$R = \{\mathbf{R}, R^N\}, \qquad R^N \in e_N \quad \text{and} \quad \mathbf{R} \in V_{N-1}. \tag{2.2}$$

For convenience, we shall frequently use the notation

$$R^N = Z \tag{2.3}$$

and

$$R = \{\mathbf{R}, Z\}. \tag{2.4}$$

Also, we introduce a euclidean metric

$$q = q(\mathbf{R}) \tag{2.5}$$

over the $(N-1)$–dimensional vector space V_{N-1}.

With respect to an admissible coordinate basis $\{e_a\}$ in V_{N-1}, we obtain the coordinate representations

$$\mathbf{R} = \{R^a\} = \{R^1, \ldots, R^{N-1}\} \tag{2.6}$$

and

$$R = \{R^p\} = \{R^a, R^N\} \equiv \{R^a, Z\}, \tag{2.7}$$

together with

$$q(\mathbf{R}) = \sqrt{r_{ab} R^a R^b}, \tag{2.8}$$

where r_{ab} are the components of a symmetric positive–definite tensor defined over V_{N-1}. The indices $(a, b, \ldots)$ and $(p, q, \ldots)$ will be specified over the ranges $(1, \ldots, N-1)$ and $(1, \ldots, N)$, respectively; vector indices are up, co–vector indices are down; repeated up–down indices are automatically summed; the notation δ^a_b will stand for the Kronecker symbol. The variables

$$w^a = R^a/Z, \quad w_a = r_{ab} w^b, \quad w = q/Z, \tag{2.9}$$

where

$$w \in (-\infty, \infty), \tag{2.10}$$

are convenient whenever $Z \neq 0$. Sometimes we shall mention the associated metric tensor

$$r_{pq} = \{r_{NN} = 1,\ r_{Na} = 0,\ r_{ab}\} \tag{2.11}$$

meaningful over the whole vector space V_N.

Given a parameter g subject to the inequality

$$-2 < g < 2, \tag{2.12}$$

we introduce the convenient notation

$$h = \sqrt{1 - \frac{1}{4}g^2}, \tag{2.13}$$

$$G = g/h, \tag{2.14}$$

$$g_+ = \frac{1}{2}g + h, \qquad g_- = \frac{1}{2}g - h, \tag{2.15}$$

$$g^+ = -\frac{1}{2}g + h, \qquad g^- = -\frac{1}{2}g - h, \tag{2.16}$$

so that

$$g_+ + g_- = g, \qquad g_+ - g_- = 2h, \tag{2.17}$$

$$g^+ + g^- = -g, \qquad g^+ - g^- = 2h, \tag{2.18}$$

$$(g_+)^2 + (g_-)^2 = 2, \tag{2.19}$$

$$(g^+)^2 + (g^-)^2 = 2, \tag{2.20}$$

and

$$g_+ \stackrel{g \to -g}{\Longleftrightarrow} -g_-, \qquad g^+ \stackrel{g \to -g}{\Longleftrightarrow} -g^-. \tag{2.21}$$

The characteristic quadratic form

$$B(g;R) = Z^2 + gqZ + q^2 \equiv \frac{1}{2}\Big[(Z + g_+q)^2 + (Z + g_-q)^2\Big] > 0 \tag{2.22}$$

is of the negative discriminant, namely

$$D_{\{B\}} = -4h^2 < 0, \tag{2.23}$$

because of Eqs. (2.12) and (2.13). Whenever $Z \neq 0$, it is also convenient to use the quadratic form

$$Q(g;w) \stackrel{\text{def}}{=} B/(Z)^2, \tag{2.24}$$

obtaining

$$Q(g;w) = 1 + gw + w^2 > 0, \tag{2.25}$$

together with the function

$$E(g;w) \stackrel{\text{def}}{=} 1 + \frac{1}{2}gw. \tag{2.26}$$

The identity

$$E^2 + h^2w^2 = Q \tag{2.27}$$

can readily be verified. In the limit $g \to 0$, the definition (2.22) degenerates to the quadratic form of the input metric tensor (2.11):

$$B\big|_{g=0} = r_{pq}R^pR^q. \tag{2.28}$$

Also

$$Q\big|_{g=0} = 1 + w^2. \tag{2.29}$$

In terms of this notation, we propose the FMF

$$K(g;R) = \sqrt{B(g;R)}\, J(g;R), \tag{2.30}$$

where

$$J(g;R) = \mathrm{e}^{\frac{1}{2}G\Phi(g;R)}, \tag{2.31}$$

$$\Phi(g;R) = \frac{\pi}{2} + \arctan\frac{G}{2} - \arctan\Big(\frac{q}{hZ} + \frac{G}{2}\Big), \qquad \text{if} \quad Z \geq 0, \tag{2.32}$$

$$\Phi(g;R) = -\frac{\pi}{2} + \arctan\frac{G}{2} - \arctan\Big(\frac{q}{hZ} + \frac{G}{2}\Big), \qquad \text{if} \quad Z \leq 0, \tag{2.33}$$

or in other convenient forms,

$$\Phi(g;R) = \frac{\pi}{2} + \arctan\frac{G}{2} - \arctan\Big(\frac{L(g;R)}{hZ}\Big), \qquad \text{if} \quad Z \geq 0, \tag{2.34}$$

$$\Phi(g;R) = -\frac{\pi}{2} + \arctan\frac{G}{2} - \arctan\Big(\frac{L(g;R)}{hZ}\Big), \qquad \text{if} \quad Z \leq 0, \tag{2.35}$$

where

$$L(g;R) = q + \frac{g}{2}Z, \tag{2.36}$$

and

$$\Phi(g;R) = \frac{\pi}{2} - \arctan\frac{hq}{A(g;R)}, \qquad \text{if} \quad Z \geq 0, \tag{2.37}$$

$$\Phi(g;R) = -\frac{\pi}{2} - \arctan\frac{hq}{A(g;R)}, \qquad \text{if} \quad Z \leq 0, \tag{2.38}$$

where

$$A(g;R) = Z + \frac{1}{2}gq. \tag{2.39}$$

This FMF has been normalized to show the handy properties

$$-\frac{\pi}{2} \le \Phi \le \frac{\pi}{2}, \tag{2.40}$$

$$\Phi = \frac{\pi}{2}, \quad \text{if} \quad q = 0 \quad \text{and} \quad Z > 0; \qquad \Phi = -\frac{\pi}{2}, \quad \text{if} \quad q = 0 \quad \text{and} \quad Z < 0. \tag{2.41}$$

We also have

$$\cot \Phi = \frac{hq}{A}, \qquad \Phi|_{Z=0} = \arctan \frac{G}{2}. \tag{2.42}$$

It is often convenient to use the indicator of sign ϵ_Z for the argument Z:

$$\epsilon_Z = 1, \quad \text{if} \quad Z > 0; \qquad \epsilon_Z = -1, \quad \text{if} \quad Z < 0; \tag{2.43}$$

Under these conditions, we call the considered space *the* $\mathcal{E}_g^{PD}$*–space*:

$$\mathcal{E}_g^{PD} = \{V_N = V_{N-1} \times e_N;\ R \in V_N;\ K(g;R);\ g\}. \tag{2.44}$$

The right–hand part of the definition (2.30) can be considered to be a function $\breve{K}$ of the arguments $\{g; q, Z\}$, such that

$$\breve{K}(g; q, Z) = K(g; R). \tag{2.45}$$

We observe that

$$\breve{K}(g; q, -Z) \ne \breve{K}(g; q, Z), \qquad \textit{unless} \quad g = 0. \tag{2.46}$$

Instead, the function $\breve{K}$ shows the property of *gZ–parity*

$$\breve{K}(-g; q, -Z) = \breve{K}(g; q, Z). \tag{2.47}$$

The $(N-1)$–space reflection invariance holds true

$$K(g; R) \stackrel{R^a \leftrightarrow -R^a}{\Leftrightarrow} K(g; R). \tag{2.48}$$

It is frequently convenient to rewrite the representation (2.30) in the form

$$K(g; R) = |Z| V(g; w), \tag{2.49}$$

whenever $Z \ne 0$, with *the generating metric function*

$$V(g; w) = \sqrt{Q(g; w)}\, j(g; w). \tag{2.50}$$

We have

$$j(g; w) = J(g; 1, w).$$

Using (2.25) and (2.31)–(2.35), we obtain

$$V' = wV/Q, \qquad V'' = V/Q^2, \tag{2.51}$$

$$(V^2/Q)' = -gV^2/Q^2, \qquad (V^2/Q^2)' = -2(g+w)V^2/Q^3, \tag{2.52}$$

$$j' = -\frac{1}{2}gj/Q, \tag{2.53}$$

and also

$$\frac{1}{2}(V^2)' = wV^2/Q, \qquad \frac{1}{2}(V^2)'' = (Q-gw)V^2/Q^2, \tag{2.54}$$

$$\frac{1}{4}(V^2)''' = -gV^2/Q^3, \tag{2.55}$$

together with

$$\Phi' = -h/Q, \tag{2.56}$$

where the prime ($'$) denotes the differentiation with respect to w.

Also,

$$(A(g;R))^2 + h^2q^2 = B(g;R) \tag{2.57a}$$

and

$$(L(g;R))^2 + h^2Z^2 = B(g;R). \tag{2.57b}$$

Sometimes it is convenient to use the function

$$E(g;w) \stackrel{\text{def}}{=} 1 + \frac{1}{2}gw. \tag{2.58}$$

The simple results for these derivatives reduce the task of computing the components of the associated FMT to an easy exercise, indeed:

$$R_p \stackrel{\text{def}}{=} \frac{1}{2}\frac{\partial K^2(g;R)}{\partial R^p}:$$

$$R_a = r_{ab}R^b\frac{K^2}{B}, \qquad R_N = (Z+gq)\frac{K^2}{B}; \tag{2.59}$$

$$g_{pq}(g;R) \stackrel{\text{def}}{=} \frac{1}{2}\frac{\partial^2 K^2(g;R)}{\partial R^p \partial R^q} = \frac{\partial R_p(g;R)}{\partial R^q}:$$

$$g_{NN}(g;R) = [(Z+gq)^2 + q^2]\frac{K^2}{B^2}, \qquad g_{Na}(g;R) = gqr_{ab}R^b\frac{K^2}{B^2}, \tag{2.60}$$

$$g_{ab}(g;R) = \frac{K^2}{B}r_{ab} - g\frac{r_{ad}R^d r_{be}R^e Z}{q}\frac{K^2}{B^2}. \tag{2.61}$$

The reciprocal tensor components are

$$g^{NN}(g;R) = (Z^2+q^2)\frac{1}{K^2}, \qquad g^{Na}(g;R) = -gqR^a\frac{1}{K^2}, \tag{2.62}$$

$$g^{ab}(g;R) = \frac{B}{K^2}r^{ab} + g(Z+gq)\frac{R^aR^b}{q}\frac{1}{K^2}. \tag{2.63}$$

The determinant of the FMT given by Eqs. (2.59)–(2.60) can readily be found in the form

$$\det(g_{pq}(g;R)) = [J(g;R)]^{2N}\det(r_{ab}) \tag{2.64}$$

which shows, on noting (2.31)–(2.33), that

$$\det(g_{pq}) > 0 \quad \textit{over all the definition range} \quad V_N \setminus 0. \tag{2.65}$$

The associated angular metric tensor

$$h_{pq} \stackrel{\text{def}}{=} g_{pq} - R_pR_q\frac{1}{K^2}$$

proves to be given by the components

$$h_{NN}(g;R) = q^2\frac{K^2}{B^2}, \qquad h_{Na}(g;R) = -Zr_{ab}R^b\frac{K^2}{B^2},$$

$$h_{ab}(g;R) = \frac{K^2}{B}r_{ab} - (gZ+q)\frac{r_{ad}R^dr_{be}R^e}{q}\frac{K^2}{B^2},$$

which entails

$$\det(h_{ab}) = \det(g_{pq})\frac{1}{V^2}.$$

The use of the components of the Cartan tensor (given explicitly in the end of the present section) leads, after rather tedious straightforward calculations, to the following simple and remarkable result.

PROPOSITION 1. *The Cartan tensor associated with the FMF* (2.30) *is of the following special algebraic form:*

$$C_{pqr} = \frac{1}{N}\left(h_{pq}C_r + h_{pr}C_q + h_{qr}C_p - \frac{1}{C_sC^s}C_pC_qC_r\right) \tag{2.66}$$

with

$$C_tC^t = \frac{N^2}{4K^2}g^2. \tag{2.67}$$

By the help of (2.65), elucidating the structure of the curvature tensor

$$S_{pqrs} \stackrel{\text{def}}{=} (C_{tqr}C_p{}^t{}_s - C_{tqs}C_p{}^t{}_r) \tag{2.68}$$

results in the simple representation

$$S_{pqrs} = -\frac{C_t C^t}{N^2}(h_{pr}h_{qs} - h_{ps}h_{qr}). \tag{2.69}$$

Inserting here (2.66), we are led to

PROPOSITION 2. *The curvature tensor of the space $\mathcal{E}_g^{PD}$ is of the special type*

$$S_{pqrs} = S^*(h_{pr}h_{qs} - h_{ps}h_{qr})/K^2 \tag{2.70}$$

with

$$S^* = -\frac{1}{4}g^2. \tag{2.71}$$

DEFINITION. FMF (2.30) introduces an $(N-1)$–dimensional indicatrix hypersurface according to the equation

$$K(g;R) = 1. \tag{2.72}$$

We call this particular hypersurface *the Finsleroid*, to be denoted as $\mathcal{F}_g^{PD}$.

Recalling the known formula $\mathcal{R} = 1 + S^*$ for the indicatrix curvature (see [5]), from (2.71) we conclude that

$$\mathcal{R}_{Finsleroid} = h^2 = 1 - \frac{1}{4}g^2, \qquad 0 < \mathcal{R}_{Finsleroid} \leq 1. \tag{2.73}$$

Geometrically, the fact that the quantity (2.70) is independent of vectors R means that the indicatrix curvature is constant. Therefore, we have arrived at

PROPOSITION 3. *The Finsleroid $\mathcal{F}_g^{PD}$ is a constant-curvature space with the positive curvature value* (2.73).

Also, on comparing between the result (2.73) and Eqs. (2.22)–(2.23), we obtain

PROPOSITION 4. *The Finsleroid curvature relates to the discriminant* (2.23) *of the input characteristic quadratic form* (2.22) *simply as*

$$\mathcal{R}_{Finsleroid} = -\frac{1}{4}D_{\{B\}}. \tag{2.74}$$

Last, we write down the explicit components of the relevant Cartan tensor

$$C_{pqr} \stackrel{\text{def}}{=} \frac{1}{2}\frac{\partial g_{pq}}{\partial R^r}:$$

$$R^N C_{NNN} = gw^3V^2Q^{-3}, \qquad R^N C_{aNN} = -gww_aV^2Q^{-3},$$

$$R^N C_{abN} = \frac{1}{2}gwV^2Q^{-2}r_{ab} + \frac{1}{2}g(1-gw-w^2)w_aw_bw^{-1}V^2Q^{-3},$$

$$R^N C_{abc} = -\frac{1}{2}gV^2Q^{-2}w^{-1}(r_{ab}w_c + r_{ac}w_b + r_{bc}w_a) + gw_aw_bw_cw^{-3}\left(\frac{1}{2}Q + gw + w^2\right)V^2Q^{-3};$$

and

$$R^N C_N{}^N{}_N = gw^3/Q^2, \qquad R^N C_a{}^N{}_N = -gww_a/Q^2,$$

$$R^N C_N{}^a{}_N = -gw(1+gw)w^a/Q^2,$$

$$R^N C_a{}^N{}_b = \frac{1}{2}gwr_{ab}/Q + \frac{1}{2}g(1-gw-w^2)w_a w_b/wQ^2,$$

$$R^N C_N{}^a{}_b = \frac{1}{2}gw\delta^a_b/Q + \frac{1}{2}g(1+gw-w^2)w^a w_b/wQ^2,$$

$$R^N C_a{}^b{}_c = -\frac{1}{2}g\left(\delta^b_a w_c + \delta^b_c w_a + (1+gw)r_{ac}w^b\right)/wQ + \frac{1}{2}g(gwQ+Q+2w^2)w_a w^b w_c/w^3Q^2.$$

The components have been calculated by the help of the formulae (2.50)–(2.53).

The use of the contractions

$$R^N C_a{}^b{}_c r^{ac} = -g\frac{w^b}{w}\frac{1+gw}{Q}\left(\frac{N-2}{2}+\frac{1}{Q}\right)$$

and

$$R^N C_a{}^b{}_c w^a w^c = -g\frac{w}{Q^2}(1+gw)w^b$$

is handy in many calculations.

Also,

$$R^N C_N = \frac{N}{2}gwQ^{-1}, \qquad R^N C_a = -\frac{N}{2}g(w_a/w)Q^{-1},$$

$$R^N C^N = \frac{N}{2}gw/V^2, \qquad R^N C^a = -\frac{N}{2}gw^a(1+gw)/wV^2,$$

$$C^N = \frac{N}{2}gwR^N K^{-2}, \qquad C^a = -\frac{N}{2}gw^a(1+gw)w^{-1}R^N K^{-2},$$

$$C_p C^p = \frac{N^2}{4K^2}g^2.$$

2.3. Quasi-euclidean map of Finsleroid

It is possible to indicate the diffeomorphism

$$\mathcal{F}^{PD}_g \overset{i_g}{\Longrightarrow} \mathcal{S}^{PD} \tag{3.1}$$

of the Finsleroid $\mathcal{F}_g^{PD} \subset V_N$ to the unit sphere $\mathcal{S}^{PD} \subset V_N$:

$$\mathcal{S}^{PD} = \{R \in \mathcal{S}^{PD} : \ S(R) = 1\}, \tag{3.2}$$

where

$$S(R) = \sqrt{r_{pq}R^pR^q} \equiv \sqrt{(R^N)^2 + r_{ab}R^aR^b} \tag{3.3}$$

is the input euclidean metric function (see (2.11)).

The diffeomorphism (3.1) can always be extended to get the diffeomorphic map

$$V_N \stackrel{\sigma_g}{\Longrightarrow} V_N \tag{3.4}$$

of the whole vector space V_N by means of the homogeneity:

$$\sigma_g \cdot (bR) = b\sigma_g \cdot R, \quad b > 0. \tag{3.5}$$

To this end it is sufficient to take merely

$$\sigma_g \cdot R = ||R|| i_g \cdot \Bigl(\frac{R}{||R||}\Bigr), \tag{3.6}$$

where

$$||R|| = K(g; R). \tag{3.7}$$

Eqs. (3.1)–(3.7) entail

$$K(g; R) = S(\sigma_g \cdot R). \tag{3.8}$$

The identity (2.57) suggests to take the map

$$\bar{R} = \sigma_g \cdot R \tag{3.9}$$

by means of the components

$$\bar{R}^p = \sigma^p(g; R) \tag{3.10}$$

with

$$\sigma^a = R^a h J(g; R), \qquad \sigma^N = A(g; R) J(g; R), \tag{3.11}$$

where $J(g; R)$ and $A(g; R)$ are the functions (2.31) and (2.39). Indeed, inserting (3.11) in (3.3) and taking into account Eqs. (2.30) and (2.57), we get the identity

$$S(\bar{R}) = K(g; R) \tag{3.12}$$

which is tantamount to the implied relation (3.8).

PROPOSITION 5. The map given explicitly by Eqs. (3.9)–(3.11) assigns *the diffeomorphism between the Finsleroid and the unit sphere* according to Eqs. (3.1)–(3.8).

Therefore, we may also call the operation (3.1) *the quasi-euclidean map of Finsleroid.*

The inverse

$$R = \mu_g \cdot \bar{R}, \qquad \mu_g = (\sigma_g)^{-1}, \tag{3.13}$$

of the transformation (3.9)–(3.11) can be presented by the components

$$R^p = \mu^p(g; \bar{R}) \tag{3.14}$$

with

$$\mu^a = \bar{R}^a / hk(g;\bar{R}), \qquad \mu^N = I(g;\bar{R})/k(g;\bar{R}), \tag{3.15}$$

where

$$k(g;\bar{R}) \stackrel{\text{def}}{=} J(g;\mu(g;\bar{R})) \tag{3.16}$$

and

$$I(g;\bar{R}) = \bar{R}^N - \frac{1}{2} G \sqrt{r_{ab}\bar{R}^a\bar{R}^b}. \tag{3.17}$$

The identity

$$\mu^p(g;\sigma(g;R)) \equiv R^p \tag{3.18}$$

can readily be verified. Notice that

$$\frac{\sqrt{r_{ab}\bar{R}^a\bar{R}^b}}{\bar{R}^N} = \frac{hq}{A(g;R)}, \qquad w^a = \frac{R^a}{R^N} = \frac{\bar{R}^a}{hI(g;\bar{R})}, \tag{3.19}$$

and

$$\sqrt{B}/Z = S/I, \qquad \sqrt{Q} = S/I. \tag{3.20}$$

The σ_g–image

$$\phi(g;\bar{R}) \stackrel{\text{def}}{=} \Phi(g;R)\big|_{R=\mu(g;\bar{R})} \tag{3.21}$$

of the function Φ described by Eqs. (2.31)–(2.42) is of a clear meaning of angle:

$$\phi(g;\bar{R}) = \operatorname{arccot}\frac{\bar{R}^N}{\sqrt{r_{ab}\bar{R}^a\bar{R}^b}} = \begin{cases} \dfrac{\pi}{2} - \arctan\dfrac{\sqrt{r_{ab}\bar{R}^a\bar{R}^b}}{\bar{R}^N}, & \text{if} \quad \bar{R}^N \geq 0; \\[2ex] -\dfrac{\pi}{2} - \arctan\dfrac{\sqrt{r_{ab}\bar{R}^a\bar{R}^b}}{\bar{R}^N}, & \text{if} \quad \bar{R}^N \leq 0; \end{cases} \tag{3.22}$$

which ranges over

$$-\frac{\pi}{2} \leq \phi \leq \frac{\pi}{2}. \tag{3.23}$$

We have

$$\phi = \frac{\pi}{2}, \quad \text{if} \quad \bar{R}^a = 0 \quad \text{and} \quad \bar{R}^N > 0; \qquad \phi = -\frac{\pi}{2}, \quad \text{if} \quad \bar{R}^a = 0 \quad \text{and} \quad \bar{R}^N < 0, \tag{3.24}$$

and also

$$\phi|_{\bar{R}^N=0} = 0. \tag{3.25}$$

Comparing Eqs. (3.16) and (2.31) shows that

$$k = \mathrm{e}^{\frac{1}{2}G\phi}. \tag{3.26}$$

The right–hand parts in (3.11) are homogeneous functions of degree 1:

$$\sigma^p(g; bR) = b\sigma^p(g; R), \quad b > 0. \tag{3.27}$$

Therefore, the identity

$$\sigma^p_s(g; R)R^s = \bar{R}^p \tag{3.28}$$

should be valid for the derivatives

$$\sigma^q_p(g; R) \overset{\text{def}}{=} \frac{\partial \sigma^q(g; R)}{\partial R^p}. \tag{3.29}$$

The simple representations

$$\sigma^N_N(g; R) = \left(B + \frac{1}{2}gqA\right)\frac{J}{B}, \tag{3.30}$$

$$\sigma^N_a(g; R) = -\frac{g(ZA - B)}{2q}\frac{Jr_{ab}R^b}{B}, \tag{3.31}$$

$$\sigma^a_N(g; R) = \frac{1}{2}gq\frac{JR^a h}{B}, \tag{3.32}$$

$$\sigma^a_b(g; R) = \left(B\delta^a_b - \frac{gr_{bc}R^c R^a Z}{2q}\right)\frac{Jh}{B}, \tag{3.33}$$

and also the determinant value

$$\det(\sigma^q_p) = h^{N-1}J^N \tag{3.34}$$

are obtained. The relations

$$\sigma^a_b R^b = JhR^a(AZ+q^2)/B, \qquad r^{cd}\sigma^a_c\sigma^b_d = J^2h^2\left[r^{ab} - g(R^aR^bZ/qB) + \frac{1}{4}g^2(R^aR^bZ^2/B^2)\right]$$

are handy in many calculations involving the coefficients $\{\sigma^q_p\}$.

Henceforth, to simplify notation, we shall use the substitution

$$t^p = \bar{R}^p. \tag{3.35}$$

Again, we can note the homogeneity

$$\mu^p(g; bt) = b\mu^p(g; t), \quad b > 0, \tag{3.36}$$

for the functions (3.15), which entails the identity

$$\mu^p_s(g;t)t^s = R^p \tag{3.37}$$

for the derivatives

$$\mu^p_q(g;t) \stackrel{\text{def}}{=} \frac{\partial\mu^p(g;t)}{\partial t^q}. \tag{3.38}$$

We find

$$\mu^N_N = 1/k(g;t) - \frac{1}{2}g\frac{m(t)I(g;t)}{k(g;t)(S(t))^2}, \qquad \mu^N_a = \frac{1}{2}g\frac{r_{ac}t^cI^*(g;t)}{k(g;t)(S(t))^2}, \tag{3.39}$$

$$\mu^a_N = -\frac{1}{2}g\frac{m(t)\,t^a}{hk(g;t)(S(t))^2}, \qquad \mu^a_b = \frac{1}{hk(g;t)}\delta^a_b + \frac{1}{2}g\frac{t^Nt^ar_{bc}t^c}{m(t)\,hk(g;t)(S(t))^2}, \tag{3.40}$$

where

$$m(t) = \sqrt{r_{ab}t^at^b}, \tag{3.41}$$

$$I^*(g;t) = hm(t) - \frac{1}{2}gt^N, \tag{3.42}$$

and

$$S(t) = \sqrt{r_{rs}t^rt^s} \equiv \sqrt{(t^N)^2 + r_{ab}t^at^b}. \tag{3.43}$$

The relations

$$\frac{\partial(1/k(g;t))}{\partial t^N} = -\frac{1}{2}g\frac{m(t)}{hk(g;t)(S(t))^2}, \qquad \frac{\partial(1/k(g;t))}{\partial t^a} = \frac{1}{2}g\frac{t^Nr_{ab}t^b}{m(t)hk(g;t)(S(t))^2}$$

are obtained.

Also

$$R_p\mu^p_q = t_q, \qquad t_p\sigma^p_q = R_q. \tag{3.44}$$

The unit vectors

$$L^p \stackrel{\text{def}}{=} \frac{t^p}{S(t)}, \qquad L_p \stackrel{\text{def}}{=} r_{pq}L^q \tag{3.45}$$

fulfil the relations

$$L^q = l^p\sigma^q_p, \qquad l^p = \mu^p_qL^q, \qquad l_p = \sigma^q_pL_q, \qquad L_p = \mu^q_pl_q, \tag{3.46}$$

where $l^p = R^p/K(g;R)$ and $l_p = g_{pq}(g;R)l^q$ are the initial Finslerian unit vectors.

Now we use the explicit formulae (2.61)–(2.62) and (3.29)–(3.32) to find the transform

$$n^{rs}(g;t) \stackrel{\text{def}}{=} \sigma^r_p\sigma^s_qg^{pq} \tag{3.47}$$

of the FMT g_{pq} under the $\mathcal{F}^{PD}_g$–induced map (3.9)–(3.11), which results in

PROPOSITION 6. *One obtains the simple representation*

$$n^{rs} = h^2 r^{rs} + \frac{1}{4} g^2 L^r L^s. \tag{3.48}$$

The covariant version reads

$$n_{rs} = \frac{1}{h^2} r_{rs} - \frac{1}{4} G^2 L_r L_s. \tag{3.49}$$

The determinant of this tensor is a constant:

$$\det(n_{rs}) = h^{2(1-N)} \det(r_{ab}). \tag{3.50}$$

Notice that

$$L^p L_p = 1, \quad n_{pq} L^q = L_p, \quad n^{pq} L_q = L^p, \quad n_{pq} L^p L^q = 1, \quad n_{pq} t^p t^q = (S(t))^2.$$

Eq. (5.47) obviously entails

$$g_{pq} = n_{rs}(g;t) \sigma_p^r \sigma_q^s. \tag{3.51}$$

2.4. Quasi-euclidean metric tensor

Let us introduce

DEFINITION. The metric tensor (3.48)–(3.49) is called *quasi-euclidean.*

DEFINITION. *The quasi-euclidean space*

$$\mathcal{Q}_N = \{V_N; n_{pq}(g;t); g\} \tag{4.1}$$

is an extension of the euclidean space $\{V_N; r_{pq}\}$ to the case $g \neq 0$.

The transformation (3.47) can be inverted to read

$$g_{pq} = \sigma_p^r \sigma_q^s n_{rs}. \tag{4.2}$$

For the angular metric tensor (see the formula going below Eq. (2.64)), from (3.46) and (4.2) we infer

$$h_{pq} = \sigma_p^r \sigma_q^s H_{rs} \frac{1}{h^2}, \tag{4.3}$$

where

$$H_{rs} \stackrel{\text{def}}{=} r_{rs} - L_r L_s \tag{4.4}$$

is the tensor showing the orthogonality property

$$L^r H_{rs} = 0. \tag{4.5}$$

One can readily find that

$$H_{rs} = h^2 (n_{rs} - L_r L_s).$$

PROPOSITION 7. The quasi-euclidean metric tensor (3.48)–(3.49) *is conformal* to the euclidean metric tensor.

Indeed, if we consider the map

$$\bar{R}^p \to \tilde{R}: \qquad \tilde{R}^p = f(g;\bar{R})\bar{R}^p/h \tag{4.6}$$

with

$$f(g;\bar{R}) = a\left(g;\ \frac{1}{2}S^2(\bar{R})\right) \tag{4.7}$$

and use the coefficients

$$k_q^p \stackrel{\text{def}}{=} \frac{\partial \tilde{R}^p}{\partial \bar{R}^q} = (f\delta_q^p + a'\bar{R}^p\bar{R}_q)/h \tag{4.8}$$

to define the tensor

$$c^{pq}(g;\tilde{R}) \stackrel{\text{def}}{=} k_r^p k_s^q n^{rs}(g;\bar{R}), \tag{4.9}$$

we find that

$$c^{pq} = f^2 r^{pq} \tag{4.10}$$

whenever

$$f = \left[\frac{1}{2}S^2(\bar{R})\right]^{\gamma/2}, \tag{4.11}$$

where

$$\gamma = h - 1 \equiv \sqrt{1 - \frac{g^2}{4}} - 1 \tag{4.12}$$

is the parameter. The proof of Proposition 7 is complete.

Let us now use the obtained quasi-euclidean metric tensor $n_{pq}(g;t)$ to construct the associated *quasi-euclidean Christoffel symbols* $N_p{}^r{}_q(g;t)$. We find consecutively:

$$n_{pq,r} \stackrel{\text{def}}{=} \frac{\partial n_{pq}}{\partial t^r} = -\frac{1}{4}G^2(H_{pr}L_q + H_{qr}L_p)/S, \tag{4.13}$$

and

$$N_p{}^r{}_q = n^{rs}N_{psq}, \qquad N_{prq} = \frac{1}{2}(n_{pr,q} + n_{qr,p} - n_{pq,r}), \tag{4.14}$$

together with

$$N_{prq}(g;t) = -\frac{1}{4}G^2 H_{pq}L_r/S, \tag{4.15}$$

which eventually yields

$$N_p{}^r{}_q(g;t) = -\frac{1}{4}G^2 L^r H_{pq}/S. \tag{4.16}$$

Comparing the representation (4.16) with the identity (4.5) shows that

$$t^p N_p{}^r{}_q = 0, \qquad N_p{}^s{}_s = 0, \qquad N_t{}^s{}_r N_p{}^t{}_q = 0. \tag{4.17}$$

Also,

$$\frac{\partial N_p{}^r{}_q}{\partial t^s} - \frac{\partial N_p{}^r{}_s}{\partial t^q} = -\frac{1}{4}G^2(H_{pq}H_s{}^r - H_{ps}H_q{}^r)/S^2. \tag{4.18}$$

Using the identities (4.17)-(4.18) in *the quasi-euclidean curvature tensor*:

$$R_p{}^r{}_{qs}(g;t) \stackrel{\text{def}}{=} \frac{\partial N_p{}^r{}_q}{\partial t^s} - \frac{\partial N_p{}^r{}_s}{\partial t^q} + N_p{}^w{}_q N_w{}^r{}_s - N_p{}^w{}_s N_w{}^r{}_q, \tag{4.19}$$

we arrive at the simple result:

$$R_{prqs}(g;t) = -\frac{1}{4}G^2(H_{pq}H_{rs} - H_{ps}H_{qr})/S^2. \tag{4.20}$$

This infers the identities

$$L^pR_{pqrs} = L^qR_{pqrs} = L^rR_{pqrs} = L^sR_{pqrs} = 0. \tag{4.21}$$

NOTE. Because of the transformation rules (3.12) and (3.47), the representation (4.20) is tantamount to Eqs. (2.69)–(2.70). Therefore *we have got another rigorous proof of Proposition* 3, *and of Eq.* (2.71), *concerning the Finsleroid curvature.*

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